Quantitative Methods

Rates of Return

Exam Focus

- Describe, compare, and interpret returns.

Assets, Instruments, and Indicators

Financial assets

- Cash
- Equity ownership
- Debt investment
- Hybrid debt/equity securities

Financial instruments

- Equity and debt securities
- Tradable standardized contracts

Assets, Instruments, and Indicators

Financial indicators

- Interest rates
- Currency exchange rates
- Market indexes

Single-Period Returns

Two sources of asset returns: price changes and cash income

Price return = $\frac{P_1 - P_0}{P_0}$

Capital distribution return = $\frac{\text{income}}{P_0}$

Total return = $\frac{P_1 - P_0 + \text{income}}{P_0}$ = price return + capital distribution return

Price, Capital Distribution, and Total Returns: Example

Calculate the price return, dividend yield, and total return for Ørsted A/S equity in 2020 given the following information:

Date	Price (per share in DKK)	Dividend (per share in DKK)
31 December 2019	689.00	9.75
31 December 2020	1,243.50	10.50

Price return = $\frac{P_1 - P_0}{P_0}$ = _____ =

Price, Capital Distribution, and Total Returns: Example (cont.)

Dividend yield (capital distribution return) = $\frac{\text{income}}{P_0}$ = = =

Total return = $\frac{P_1 - P_0 + \text{income}}{P_0}$ = =

Alternatively, total return =

Terminology

- Actual return = ex post return
- Expected return = ex ante return

- Realized returns = capital distributions received and proceeds from selling an investment
- Unrealized returns are based on current market price of an asset still held by investor

Returns Across Multiple Periods

Arithmetic mean = $\frac{(R_1 + R_2 + R_3 + ... + R_n)}{n}$

- Arithmetic mean does *not* reflect multiperiod compounding
  → Most appropriate for single-period returns

Geometric mean = $\sqrt[n]{(1+R_1)(1+R_2)(1+R_3)...(1+R_n)} - 1$

- Geometric mean *does* reflect multiperiod compounding
  → Most appropriate for average return over multiple periods

Returns Across Multiple Periods

Arithmetic-geometric mean inequality principle

- Geometric mean is always less than or equal to the arithmetic mean
- Difference increases as variability of observations increases

Returns Across Multiple Periods: Example

Calculate the arithmetic and geometric average annual total return for Ørsted A/S equity for 2017–2023, given the following information:

Date	Total Return (in %)
31 December 2017	28.81%
31 December 2018	31.30%
31 December 2019	60.37%
31 December 2020	82.00%
31 December 2021	–31.91%
31 December 2022	–22.92%
31 December 2023	–38.57%

Returns Across Multiple Periods: Example (cont.)

Arithmetic average annual total return =

$$\frac{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}{}$$

=

Geometric average annual total return =

$$\sqrt{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad} - 1$$

=

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Compounding Periods and Annualized Returns

$$\text{annualized return} = (1 + \text{return during the period})^{(365/\text{days})} - 1$$

$$\text{annualized return} = (1 + \text{return during the period})^{c} - 1$$

where:
c = number of periods that make up a year (e.g., weeks, months, quarters)

Annualizing returns helps comparability but assumes returns are repeatable.

Annualized Returns: Example 1

An investor evaluating the returns of three recently formed exchange-traded funds gathers the following information. Assuming that a year has 365 days, or 52 weeks, or 12 months, the ETF with the highest annualized rate of return is:

A. ETF 1.
B. ETF 2.
C. ETF 3.

ETF	Time Since Inception	Return Since Inception (%)
1	146 days	4.61
2	5 weeks	1.10
3	15 months	14.35

Annualized Returns: Example 1 (cont.)

$$\text{annualized return} = (1 + \text{return during the period})^{(365/\text{days})} - 1$$

Annualized return for ETF 1 =

Annualized return for ETF 2 =

Annualized return for ETF 3 =

The ETF with the highest annualized rate of return is ETF .

Annualized Returns: Example 2

Annualized returns are calculated by:

A. adding all periodic returns within a year.

B. dividing the period returns by the number of periods in a year.

C. raising the sum of 1 plus the geometric average periodic return to the power of the number of periods in a year, then subtracting 1.

-1

Continuous Compounding

$$R_{cc} = \ln(1 + \text{return during the period}) = \ln(\frac{\text{ending value}}{\text{beginning value}})$$

Continuously compounded returns are additive for multiple periods.

Continuous Compounding: Example 1

What are the continuously compounded price returns for Ørsted A/S equity for the semi-annual period 1 January 2020–30 June 2020, given the following information?

A. 10.51%.

B. 13.64%.

C. 24.54%.

$$R_{cc} = \ln\left(\frac{\text{ending value}}{\text{beginning value}}\right)$$

=

Date	Price (per share in DKK)
31 December 2019	689.0
31 January 2020	736.0
28 February 2020	695.4
31 March 2020	666.4
30 April 2020	688.4
29 May 2020	786.8
30 June 2020	765.4

-1

Continuous Compounding: Example 2

A stock experiences a continuously compounded return of 10% for the first half of the year and 40% for the second half. Its continuously compounded return for the full year is *closest* to:

A. 50.00%.

B. 54.00%.

C. 56.25%.

-1

Solutions

Price, Capital Distribution, and Total Returns: Example

Calculate the price return, dividend yield, and total return, for Ørsted A/S equity in 2020 given the following information:

Date	Price (per share in DKK)	Dividend (per share in DKK)
31 December 2019	689.00	9.75
31 December 2020	1,243.50	10.50

$$\text{Price return} = \frac{P_1 - P_0}{P_0} = \frac{1{,}243.50 - 689.00}{689.00} = 80.48\%$$

Price, Capital Distribution, and Total Returns: Example (cont.)

$$\text{Dividend yield (capital distribution return)} = \frac{\text{income}}{P_0} = \frac{10.50}{689.00} = 1.52\%$$

$$\text{Total return} = \frac{P_1 - P_0 + \text{income}}{P_0} = \frac{1{,}243.50 - 689.00 + 10.50}{689.00} = 82.00\%$$

Alternatively, total return = 80.48% + 1.52% = 82.00%

Returns Across Multiple Periods: Example (cont.)

Arithmetic average annual total return =

$$\frac{(0.2881 + 0.3130 + 0.6037 + 0.8200 + -0.3191 + -0.2292 + -0.3857)}{7}$$

$$= 0.1558 = 15.58\%$$

Geometric average annual total return =

$$\sqrt[7]{(1+0.2881)(1+0.313)(1+0.6037)(1+0.82)(1-0.3191)(1-0.2292)(1-0.3857)} - 1$$

$$= 0.0686 = 6.86\%$$

Annualized Returns: Example 1

An investor evaluating the returns of three recently formed exchange-traded funds gathers the following information. Assuming that a year has 365 days, or 52 weeks, or 12 months, the ETF with the highest annualized rate of return is:

A. ETF 1.
B. ETF 2.
C. ETF 3.

ETF	Time Since Inception	Return Since Inception (%)
1	146 days	4.61
2	5 weeks	1.10
3	15 months	14.35

Annualized Returns: Example 1 (cont.)

$$\text{annualized return} = (1 + \text{return during the period})^{(365/\text{days})} - 1$$

Annualized return for ETF 1 = $1.0461^{(365/146)} - 1 = 11.93\%$

Annualized return for ETF 2 = $1.0110^{(52/5)} - 1 = 12.05\%$

Annualized return for ETF 3 = $1.1435^{(12/15)} - 1 = 11.32\%$

The ETF with the highest annualized rate of return is ETF 2.

Annualized Returns: Example 2

Annualized returns are calculated by:

A. adding all periodic returns within a year.

B. dividing the period returns by the number of periods in a year.

C. raising the sum of 1 plus the geometric average periodic return to the power of the number of periods in a year, then subtracting 1.

Continuous Compounding: Example 1

What are the continuously compounded price returns for Ørsted A/S equity for the semi-annual period 1 January 2020-30 June 2020, given the following information?

A. 10.51%.

B. 13.64%.

C. 24.54%.

Date	Price (per share in DKK)
31 December 2019	689.0
31 January 2020	736.0
28 February 2020	695.4
31 March 2020	666.4
30 April 2020	688.4
29 May 2020	786.8
30 June 2020	765.4

$$R_{cc} = \ln\left(\frac{\text{ending value}}{\text{beginning value}}\right)$$

$$= \ln\frac{765.4}{689.0} = 10.52\%$$

Continuous Compounding: Example 2

A stock experiences a continuously compounded return of 10% for the first half of the year and 40% for the second half. Its continuously compounded return for the full year is *closest* to:

A. 50.00%.

B. 54.00%.

C. 56.25%.

Components and Measures of Return

Exam Focus

- Describe, compare, and interpret required rates of return, risk-free rates, risk premia, and inflation.

Interest Rates

- Required rate of return
- Discount rate
- Opportunity cost of current consumption

Interest rates consist of a risk-free rate and a risk premium.

Risk-Free Rates of Return

(1 + nominal risk-free rate) = (1 + real risk-free rate)(1 + inflation premium)

$$\text{real risk-free rate} = \frac{1 + \text{nominal risk} - \text{free rate}}{1 + \text{inflation rate}} - 1$$

Approximated to:

real risk-free rate ≈ nominal risk-free rate – expected inflation rate

Risk Premia

Risk premiums (above the nominal risk-free rate) include different types of risk, all increasing the required rate of return.

Typical **bond** risk premiums:

- Default risk premium
- Liquidity risk premium
- Maturity risk premium

Typical **stock** risk premiums:

- Size risk premium
- Value risk premium

Equity Returns: Example

An analyst observes the following historic geometric average returns. The average inflation rate was 2.1% during that time.

Asset Class	Geometric Average Return (%)
Equities	8.0
Corporate bonds	6.5
Treasury bills	2.5

1. The real rate of return for equities is:

A. 5.01%.

B. 5.78%.

C. 5.90%.

-3

Equity Returns: Example (cont.)

An analyst observes the following historic geometric average returns. The average inflation rate was 2.1% during that time.

Asset Class	Geometric Average Return (%)
Equities	8.0
Corporate bonds	6.5
Treasury bills	2.5

2. The risk premium for equities is *closest* to:

A. 3.20%.

B. 5.50%.

C. 5.78%.

-2

Common Measures of Return

Excess return = return above (below) benchmark return or reference rate

Gross return = total return – transaction costs

Net return = gross return – management fees and administrative costs

Gross and Net Return: Example

An investment fund, specialized in environmentally sustainable and socially responsible investments, reported a total annual return of –5% before costs. The fund's total assets were valued at CHF 500 million. The fund incurs transaction costs of 0.05%, a management fee of 1.00%, and administrative expenses amounting to 0.50%—all as a percentage of its total asset value. The management fee is assessed at the assets at the end of the year.

1. The fund's **gross** return is *closest* to:

A. –4.95%.

B. –5.05%.

C. –6.55%.

-2

Gross and Net Return: Example (cont.)

An investment fund, specialized in environmentally sustainable and socially responsible investments, reported a total annual return of –5% before costs. The fund's total assets were valued at CHF 500 million. The fund incurs transaction costs of 0.05%, a management fee of 1.00%, and administrative expenses amounting to 0.50%—all as a percentage of its total asset value. The management fee is assessed at the assets at the end of the year.

2. The fund's **net** return is *closest* to:

A. –6.55%.

B. –6.05%.

C. –5.05%.

-2

Common Measures of Return: Tax

Pretax nominal return = return before paying taxes

After-tax nominal return = return after tax liability

- Long-term capital gains are often taxed at lower rates than capital distributions (dividends/interest) or short-term capital gains.
- Capital gains usually taxed when realized (i.e., when investment is sold).

After-tax return = (price return)(1 – capital gains tax rate) +
(capital distribution return)(1 – capital distribution tax rate)

Tax Impact on Returns: Example

In 2020, an investor purchased 1,000 shares of Ørsted A/S at a price of DKK 689.00 per share. By the end of the year, the price had risen to DKK 1,243.50 per share. Additionally, the investor received a dividend of DKK 10.50 per share during the year. The investor sold all their shares at the end of the year. The capital gains tax rate is 42%. The capital distributions, such as dividends, are taxed at the income tax rate of 27%.

1. For tax purposes, the investor's realized capital gain is:

A. DKK 543,500.

B. DKK 554,500.

C. DKK 555,000.

Realized capital gain =

=

-2

Tax Impact on Returns: Example (cont.)

In 2020, an investor purchased 1,000 shares of Ørsted A/S at a price of DKK 689.00 per share. By the end of the year, the price had risen to DKK 1,243.50 per share. Additionally, the investor received a dividend of DKK 10.50 per share during the year. The investor sold all their shares at the end of the year. The capital gains tax rate is 42%. The capital distributions, such as dividends, are taxed at the income tax rate of 27%.

2. The investor must pay a capital gains tax of:

A. DKK 149,715.

B. DKK 232,890.

C. DKK 321,610.

Capital gains tax =

=

-1

Tax Impact on Returns: Example (cont.)

In 2020, an investor purchased 1,000 shares of Ørsted A/S at a price of DKK 689.00 per share. By the end of the year, the price had risen to DKK 1,243.50 per share. Additionally, the investor received a dividend of DKK 10.50 per share during the year. The investor sold all their shares at the end of the year. The capital gains tax rate is 42%. The capital distributions, such as dividends, are taxed at the income tax rate of 27%.

3. The total after-tax return on the investment is:

A. 46.68%.

B. 47.56%.

C. 47.79%.

Tax Impact on Returns: Example (cont.)

After-tax return = (price return)(1 – capital gains tax rate) +
(capital distribution return)(1 – capital distribution tax rate) where:

Price return =

Capital distribution return =

After-tax return =

Leveraged Returns

$$\text{Leveraged return} = \frac{r_p(V_D + V_E) - r_D v_D}{V_E}$$

- Leverage uses borrowed funds (or derivatives) to increase the amount that can be invested in an asset.
- Leveraged returns combine (1) the return on the total investment and (2) the cost of borrowing, expressed as a percentage of the investor's cash investment.
- Leverage magnifies gains but also magnifies losses.

Leveraged Returns: Example

Leverage can significantly affect investment returns. An investor bought 1,000 shares of Ørsted A/S at the start of 2022 for DKK 835.20 per share. The investor financed the purchase by borrowing 50% of the purchase price at a 7% annual rate of interest. The investor held onto the 1,000 shares throughout the year, received the DKK 12.50 dividend per share, sold the shares at the end of the year for DKK 631.30 per share, and repaid the loan.

1. Determine V_E and V_D and the financing cost associated with V_D, r_d.

$V_E = 1{,}000 \times \quad \times$

$V_D =$

$r_d =$

-4

Leveraged Returns: Example (cont.)

Leverage can significantly affect investment returns. An investor bought 1,000 shares of Ørsted A/S at the start of 2022 for DKK 835.20 per share. The investor financed the purchase by borrowing 50% of the purchase price at a 7% annual rate of interest. The investor held onto the 1,000 shares throughout the year, received the DKK 12.50 dividend per share, sold the shares at the end of the year for DKK 631.30 per share, and repaid the loan.

2. Determine total investment return, r_p.

$$\text{Total return} = \frac{P_1 - P_0 + \text{income}}{P_0} =$$

Leveraged Returns: Example (cont.)

Leverage can significantly affect investment returns. An investor bought 1,000 shares of Ørsted A/S at the start of 2022 for DKK 835.20 per share. The investor financed the purchase by borrowing 50% of the purchase price at a 7% annual rate of interest. The investor held onto the 1,000 shares throughout the year, received the DKK 12.50 dividend per share, sold the shares at the end of the year for DKK 631.30 per share, and repaid the loan.

3. Determine the leveraged return, r_{LP}.

Leveraged return =

$$\frac{r_p(V_D + V_E) - r_D v_D}{V_E} = \_\_\_\_\_ =$$

Solutions

Equity Returns: Example

An analyst observes the following historic geometric average returns. The average inflation rate was 2.1% during that time.

Asset Class	Geometric Average Return (%)
Equities	8.0
Corporate bonds	6.5
Treasury bills	2.5

1. The real rate of return for equities is:

A. 5.01%.

B. 5.78%.

C. 5.90%.

$$\text{real rate} = \frac{1+\text{nominal rate}}{1+\text{inflation rate}} - 1 = \frac{1.08}{1.021} - 1 = 5.78\%$$

Equity Returns: Example (cont.)

An analyst observes the following historic geometric average returns. The average inflation rate was 2.1% during that time.

Asset Class	Geometric Average Return (%)
Equities	8.0
Corporate bonds	6.5
Treasury bills	2.5

2. The risk premium for equities is *closest* to:

A. 3.20%.
B. 5.50%.
C. 5.78%.

risk premium = equity return – nominal risk-free rate
= 8.0% – 2.5% = 5.5%

Gross and Net Return: Example

An investment fund, specialized in environmentally sustainable and socially responsible investments, reported a total annual return of –5% before costs. The fund's total assets were valued at CHF 500 million. The fund incurs transaction costs of 0.05%, a management fee of 1.00%, and administrative expenses amounting to 0.50%—all as a percentage of its total asset value. The management fee is assessed at the assets at the end of the year.

1. The fund's **gross** return is *closest* to:

A. –4.95%.
B. –5.05%.
C. –6.55%.

Gross return = total return – transaction costs
= –5% – 0.05% = –5.05%

Gross and Net Return: Example (cont.)

An investment fund, specialized in environmentally sustainable and socially responsible investments, reported a total annual return of –5% before costs. The fund's total assets were valued at CHF 500 million. The fund incurs transaction costs of 0.05%, a management fee of 1.00%, and administrative expenses amounting to 0.50%—all as a percentage of its total asset value. The management fee is assessed at the assets at the end of the year.

2. The fund's **net** return is *closest* to:

A. –6.55%.
B. –6.05%.
C. –5.05%.

Net return = gross return – mgmt. fees and admin costs
= –5.05% – 1.00% – 0.50% = –6.55%

Tax Impact on Returns: Example

In 2020, an investor purchased 1,000 shares of Ørsted A/S at a price of DKK 689.00 per share. By the end of the year, the price had risen to DKK 1,243.50 per share. Additionally, the investor received a dividend of DKK 10.50 per share during the year. The investor sold all their shares at the end of the year. The capital gains tax rate is 42%. The capital distributions, such as dividends, are taxed at the income tax rate of 27%.

1. For tax purposes, the investor's realized capital gain is:

A. DKK 543,500.
B. DKK 554,500.
C. DKK 555,000.

Realized capital gain = (1,243.50 – 689.00) × 1,000
= DKK 554,500

Tax Impact on Returns: Example (cont.)

In 2020, an investor purchased 1,000 shares of Ørsted A/S at a price of DKK 689.00 per share. By the end of the year, the price had risen to DKK 1,243.50 per share. Additionally, the investor received a dividend of DKK 10.50 per share during the year. The investor sold all their shares at the end of the year. The capital gains tax rate is 42%. The capital distributions, such as dividends, are taxed at the income tax rate of 27%.

2. The investor must pay a capital gains tax of:

A. DKK 149,715.

B. DKK 232,890.

C. DKK 321,610.

Capital gains tax = DKK 554,500 × 42%
= DKK 232,890

Tax Impact on Returns: Example (cont.)

In 2020, an investor purchased 1,000 shares of Ørsted A/S at a price of DKK 689.00 per share. By the end of the year, the price had risen to DKK 1,243.50 per share. Additionally, the investor received a dividend of DKK 10.50 per share during the year. The investor sold all their shares at the end of the year. The capital gains tax rate is 42%. The capital distributions, such as dividends, are taxed at the income tax rate of 27%.

3. The total after-tax return on the investment is:

A. 46.68%.

B. 47.56%.

C. 47.79%.

Tax Impact on Returns: Example (cont.)

After-tax return = (price return)(1 – capital gains tax rate) +
(capital distribution return)(1 – capital distribution tax rate) where:

$$\text{Price return} = \frac{1{,}243.50 - 689.00}{689.00} = 80.48\%$$

$$\text{Capital distribution return} = \frac{10.50}{689.00} = 1.52\%$$

After-tax return = (80.48%)(1 – 0.42) + (1.52%)(1 – 0.27) = 47.79%

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Leveraged Returns: Example

Leverage can significantly affect investment returns. An investor bought 1,000 shares of Ørsted A/S at the start of 2022 for DKK 835.20 per share. The investor financed the purchase by borrowing 50% of the purchase price at a 7% annual rate of interest. The investor held onto the 1,000 shares throughout the year, received the DKK 12.50 dividend per share, sold the shares at the end of the year for DKK 631.30 per share, and repaid the loan.

1. Determine V_E and V_D and the financing cost associated with V_D, r_d.

V_E = 1,000 × DKK 835.20 × 50% = DKK 417,600

V_D = DKK 417,600

r_d = 7%

Leveraged Returns: Example (cont.)

Leverage can significantly affect investment returns. An investor bought 1,000 shares of Ørsted A/S at the start of 2022 for DKK 835.20 per share. The investor financed the purchase by borrowing 50% of the purchase price at a 7% annual rate of interest. The investor held onto the 1,000 shares throughout the year, received the DKK 12.50 dividend per share, sold the shares at the end of the year for DKK 631.30 per share, and repaid the loan.

2. Determine total investment return, r_p.

$$\text{Total return} = \frac{P_1 - P_0 + \text{income}}{P_0} = \frac{631.30 - 835.20 + 12.50}{835.20} = -22.92\%$$

Leveraged Returns: Example (cont.)

Leverage can significantly affect investment returns. An investor bought 1,000 shares of Ørsted A/S at the start of 2022 for DKK 835.20 per share. The investor financed the purchase by borrowing 50% of the purchase price at a 7% annual rate of interest. The investor held onto the 1,000 shares throughout the year, received the DKK 12.50 dividend per share, sold the shares at the end of the year for DKK 631.30 per share, and repaid the loan.

3. Determine the leveraged return, r_{LP}.

Leveraged return =

$$\frac{r_p(V_D + V_E) - r_D v_D}{V_E} = \frac{-22.92\%(417{,}600 + 417{,}600) - (7\% \times 417{,}600)}{417{,}600} = -52.84\%$$