

Prerequisite Reading 1: Interest Rates, Present Value, and Future Value

This reading covers time value of money concepts and applications. Procedures are presented for calculating the future value and present value of a single cash flow, an annuity, and a series of uneven cash flows. The impact of different compounding periods is examined, along with the procedures for solving for other variables in time value of money problems. Your main objective in this reading is to master time value of money mechanics (i.e., learn how to crunch the numbers).

Professor's Note

Throughout the problems in this review, rounding differences may occur between the use of different calculators or techniques presented in this document. Do not panic if you are a few cents off in your calculations.

MODULE 1.1: INTEREST RATES

The concept of **compound interest** or **interest on interest** is deeply embedded in time value of money (TVM) procedures. When an investment is subjected to compound interest, the growth in the value of the investment from period to period reflects not only the interest earned on the original principal amount but also on the interest earned on the previous period's interest earnings—the interest on interest.

TVM applications frequently call for determining the **future value (FV)** of an investment's cash flows as a result of the effects of compound interest. Computing FV involves projecting the cash flows forward, on the basis of an appropriate compound interest rate, to the end of the investment's life. The computation of the **present value (PV)** works in the opposite direction—it brings the cash flows from an investment back to the beginning of the investment's life based on an appropriate compound rate of return.

Being able to measure the PV and/or FV of an investment's cash flows becomes useful when comparing investment alternatives because the value of the investment's cash flows must be measured at some common point in time, typically at the end of the investment horizon (FV) or at the beginning of the investment horizon (PV).

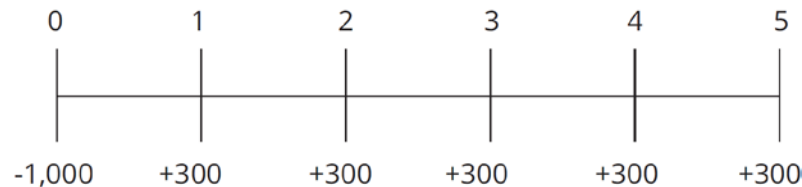
Time Lines

Drawing a time line before you start to solve a TVM problem is always a good idea. A **time line** is simply a diagram of the cash flows associated with a TVM problem. A cash flow that occurs in the present (today) is put at time zero. Cash outflows (payments) are given a negative sign, and cash inflows (receipts) are given a positive sign. Once the cash flows are assigned to a time line, they may

be moved to the beginning of the investment period to calculate the PV through a process called **discounting** or to the end of the period to calculate the FV using a process called **compounding**.

Figure 1.1 illustrates a time line for an investment that costs \$1,000 today (outflow) and will return a stream of cash payments (inflows) of \$300 per year at the end of each of the next five years.

Figure 1.1: Time Line



Please recognize that the cash flows occur at the end of the period depicted on the time line. Furthermore, note that the end of one period is the same as the beginning of the next period. For example, the end of the second year ($t = 2$) is the same as the beginning of the third year, so a cash flow at the beginning of Year 3 appears at time $t = 2$ on the time line.

LOS 1.a: Interpret interest rates as required rates of return, discount rates, or opportunity costs.

Interest rates are our measure of the time value of money, although risk differences in financial securities lead to differences in their equilibrium interest rates. Equilibrium interest rates are the **required rate of return** for a particular investment, in the sense that the market rate of return is the return that investors and savers require to get them to willingly lend their funds. Interest rates are also referred to as **discount rates** and, in fact, the terms are often used interchangeably. If an individual can borrow funds at an interest rate of 10%, then that individual should *discount* payments to be made in the future at that rate in order to get their equivalent value in current dollars or other currencies. Finally, we can also view interest rates as the **opportunity cost** of current consumption. If the market rate of interest on 1-year securities is 5%, earning an additional 5% is the opportunity forgone when current consumption is chosen rather than saving (postponing consumption).

LOS 1.b: Explain an interest rate as the sum of a real risk-free rate and premiums that compensate investors for bearing distinct types of risk.

The **real risk-free rate** of interest is a theoretical rate on a single-period loan that has no expectation of inflation in it. When we speak of a real rate of return, we are referring to an investor's increase in purchasing power (after adjusting for inflation). Since expected inflation in future periods is not zero, the rates we observe on U.S. Treasury bills (T-bills), for example, are risk-free rates but not *real* rates of return. T-bill rates are *nominal risk-free rates* because they contain an *inflation premium*. The approximate relation here is:

$$\text{nominal risk-free rate} = \text{real risk-free rate} + \text{expected inflation rate}$$

Securities may have one or more **types of risk**, and each added risk increases the required rate of return on the security. These types of risk are:

- **Default risk.** The risk that a borrower will not make the promised payments in a timely manner.
- **Liquidity risk.** The risk of receiving less than fair value for an investment if it must be sold for cash quickly.
- **Maturity risk.** As we will cover in detail in the section on debt securities, the prices of longer-term bonds are more volatile than those of shorter-term bonds. Longer maturity bonds have more maturity risk than shorter-term bonds and require a maturity risk premium.

Each of these risk factors is associated with a risk premium that we add to the nominal risk-free rate to adjust for greater default risk, less liquidity, and longer maturity relative to a very liquid, short-term, default risk-free rate such as that on T-bills. We can write:

$$\begin{aligned} \text{nominal rate of interest} &= \text{nominal risk-free rate} \\ &+ \text{default risk premium} \\ &+ \text{liquidity premium} \\ &+ \text{maturity risk premium} \end{aligned}$$

MODULE QUIZ 1.1

1. An interest rate is *best* interpreted as a:
 - A. discount rate or a measure of risk.
 - B. measure of risk or a required rate of return.
 - C. required rate of return or the opportunity cost of consumption.
2. An interest rate from which the inflation premium has been subtracted is known as a:
 - A. real interest rate.
 - B. risk-free interest rate.
 - C. real risk-free interest rate.

MODULE 1.2: CALCULATING PRESENT VALUE AND FUTURE VALUE

LOS 1.c: Calculate and interpret the future value (FV) and present value (PV) of a single sum of money, an ordinary annuity, an annuity due, a perpetuity (PV only), and a series of unequal cash flows.

Future Value of a Single Sum

Future value is the amount to which a current deposit will grow over time when it is placed in an account paying compound interest. The FV, also called the compound value, is simply an example of compound interest at work.

The formula for the FV of a *single* cash flow is:

$$FV = PV(1+I/Y)^N$$

where:

PV = amount of money invested today (the present value)

I/Y = rate of return per compounding period

N = total number of compounding periods

In this expression, the investment involves a single cash outflow, PV, which occurs today, at $t = 0$ on the time line. The single sum FV formula will determine the value of an investment at the end of N compounding periods, given that it can earn a fully compounded rate of return, I/Y, over all of the periods.

The factor $(1 + I/Y)^N$ represents the compounding rate on an investment and is frequently referred to as the **future value factor**, or the **future value interest factor**, for a single cash flow at I/Y over N compounding periods.

EXAMPLE: FV of a single sum

Calculate the FV of a \$200 investment at the end of two years if it earns an annually compounded rate of return of 10%.

Answer:

To solve this problem with your calculator, input the relevant data and compute FV.

$N = 2$; $I/Y = 10$; $PV = -200$; $CPT \rightarrow FV = \$242$

Professor's Note

Note the negative sign on PV. This is not necessary, but it makes the FV come out as a positive number. If you enter PV as a positive number, ignore the negative sign that appears on the FV.

This relatively simple problem could also be solved using the following equation:

$$FV = 200(1 + 0.10)^2 = \$242$$

On the TI calculator, enter $1.10 [x^2] \times 200 [=]$.

Present Value of a Single Sum

The PV of a single sum is today's value of a cash flow that is to be received at some point in the future. In other words, it is the amount of money that must be invested today, at a given rate of return over a given period of time, in order to end up with a specified FV. As previously mentioned, the process for finding the PV of a cash flow is known as *discounting* (i.e., future cash flows are "discounted" back to the present). The interest rate used in the discounting process is commonly referred to as the **discount rate** but may also be referred to as the **opportunity cost**, **required rate of return**, and the **cost of capital**. Whatever you want to call it, it represents the annual compound rate of return that can be earned on an investment.

The relationship between PV and FV can be seen by examining the FV expression stated earlier. Rewriting the FV equation in terms of PV, we get:

$$PV = FV_x \left[\frac{1}{\left(1 + \frac{1}{Y}\right)^N} \right] = \frac{FV}{\left(1 + \frac{1}{Y}\right)^N}$$

Note that for a single future cash flow, PV is always less than the FV whenever the discount rate is positive.

The quantity $1/(1 + I/Y)^N$ in the PV equation is frequently referred to as the **present value factor**, **present value interest factor**, or **discount factor** for a single cash flow at I/Y over N compounding periods.

EXAMPLE: PV of a single sum

Given a discount rate of 10%, calculate the PV of a \$200 cash flow that will be received in two years.

Answer:

To solve this problem, input the relevant data and compute PV.

$N = 2$; $I/Y = 10$; $FV = 200$; CPT \rightarrow $PV = -\$165.29$ (ignore the sign)

Professor's Note

With single sum PV problems, you can either enter FV as a positive number and ignore the negative sign on PV or enter FV as a negative number.

This relatively simple problem could also be solved using the following PV equation:

$$PV = \frac{200}{(1 + 0.10)^2} = \$165.29$$

On the TI, enter 1.10 [y^x] 2 [=] [1/x] [\times] 200 [=].

The PV computed here implies that at a rate of 10%, an investor will be indifferent between \$200 in two years and \$165.29 today. Put another way, \$165.29 is the amount that must be invested today at a 10% rate of return in order to generate a cash flow of \$200 at the end of two years.

Annuities

An **annuity** is a stream of *equal cash flows* that occurs at *equal intervals* over a given period. Receiving \$1,000 per year at the end of each of the next eight years is an example of an annuity. There are two types of annuities: **ordinary annuities** and **annuities due**. The *ordinary annuity* is the most common type of annuity. It is characterized by cash flows that occur at the *end* of each compounding period. This is a typical cash flow pattern for many investment and business finance applications. The other type of annuity is called an *annuity due*, where payments or receipts occur at the beginning of each period (i.e., the first payment is today at $t = 0$).

Computing the FV or PV of an annuity with your calculator is no more difficult than it is for a single cash flow. You will know four of the five relevant variables and solve for the fifth (either PV or FV). The difference between single sum and annuity TVM problems is that instead of solving for the PV or FV of a single cash flow, we solve for the PV or FV of a stream of equal periodic cash flows,

where the size of the periodic cash flow is defined by the payment (PMT) variable on your calculator.

EXAMPLE: FV of an ordinary annuity

What is the future value of an ordinary annuity that pays \$200 per year at the end of each of the next three years, given the investment is expected to earn a 10% rate of return?

Answer:

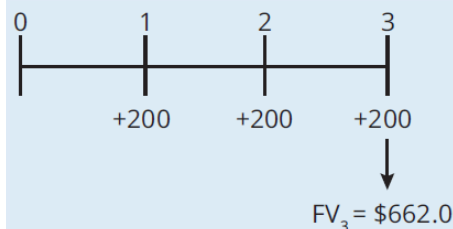
This problem can be solved by entering the relevant data and computing FV.

$N = 3$; $I/Y = 10$; $PMT = -200$; $CPT \rightarrow FV = \$662$

Implicit here is that $PV = 0$; clearing the TVM functions sets both PV and FV to zero.

The time line for the cash flows in this problem is depicted in the following figure:

FV of an Ordinary Annuity



As indicated here, the sum of the compounded values of the individual cash flows in this 3-year ordinary annuity is \$662. Note that the annuity payments themselves amounted to \$600, and the balance is the interest earned at the rate of 10% per year.

To find the PV of an ordinary annuity, we use the future cash flow stream, PMT, that we used with FV annuity problems, but we discount the cash flows back to the present (time $t = 0$) rather than compounding them forward to the terminal date of the annuity.

Here again, the PMT variable is a *single* periodic payment, *not* the total of all the payments (or deposits) in the annuity. The PVA_0 measures the collective PV of a stream of equal cash flows received at the end of each compounding period over a stated number of periods, N , given a specified rate of return, I/Y . The following examples illustrate how to determine the PV of an ordinary annuity using a financial calculator:

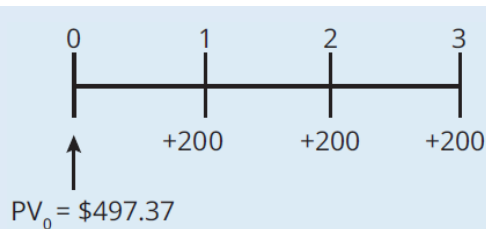
EXAMPLE: PV of an ordinary annuity

What is the PV of an annuity that pays \$200 per year at the end of each of the next three years, given a 10% discount rate?

Answer:

The payments occur at the end of the year, so this annuity is an ordinary annuity. To solve this problem, enter the relevant information and compute PV.

$N = 3$; $I/Y = 10$; $PMT = -200$; $FV = 0$; $CPT \rightarrow PV = \$497.37$



The \$497.37 computed here represents the amount of money that an investor would need to invest *today* at a 10% rate of return to generate three end-of-year cash flows of \$200 each.

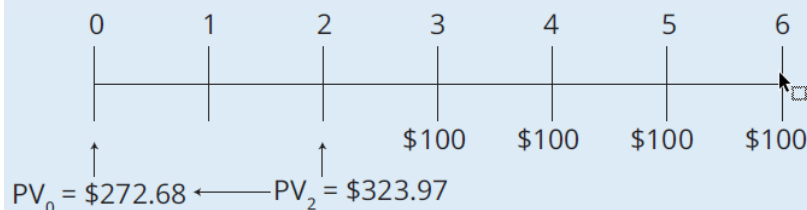
EXAMPLE: PV of an ordinary annuity beginning later than $t = 1$

What is the present value of four \$100 end-of-year payments if the first payment is to be received three years from today and the appropriate rate of return is 9%?

Answer:

The time line for this cash flow stream is shown in the following figure:

PV of an Annuity Beginning at $t = 3$



Step 1:

Find the present value of the annuity as of the end of year 2 (PV_2).

Input the relevant data and solve for PV_2 .

$N = 4$; $I/Y = 9$; $PMT = -100$; $FV = 0$; $CPT \rightarrow PV = PV_2 = \323.97

Step 2:

Find the present value of PV_2 .

Input the relevant data and solve for PV_0 .

$N = 2$; $I/Y = 9$; $PMT = 0$; $FV = -323.97$; $CPT \rightarrow PV = PV_0 = \272.68

In this solution, the annuity was treated as an ordinary annuity. The PV was computed one period before the first payment, and we discounted $PV_2 = \$323.97$ over two years. We need to stress this important point. The PV annuity function on your calculator set in “END” mode gives you the value *one period before the annuity begins*. Although the annuity begins at $t = 3$, we discounted the result for only two periods to get the present ($t = 0$) value.

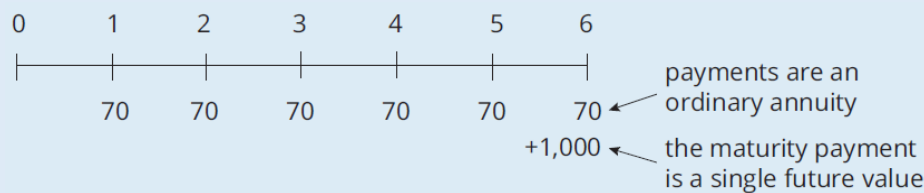
EXAMPLE: PV of a bond’s cash flows

A bond will make coupon interest payments of 70 euros (7% of its face value) at the end of each year and will also pay its face value of 1,000 euros at maturity in six years. If the appropriate discount rate is 8%, what is the present value of the bond’s promised cash flows?

Answer:

The six annual coupon payments of 70 euros each can be viewed as an ordinary annuity. The maturity value of 1,000 euros is the future value of the bond at the time the last coupon payment is made. On a time line, the promised payment stream is as shown below.

Cash Flows for a 6-Year, 7%, 1,000 Euro Bond



The PV of the bond's cash flows can be broken down into the PV of a 6-payment ordinary annuity, plus the PV of a 1,000 euro lump sum to be received six years from now.

The calculator solution is:

$N = 6$; $PMT = 70$; $I/Y = 8$; $FV = 1,000$; $CPT \rightarrow PV = -953.77$

With a yield to maturity of 8%, the value of the bond is 953.77 euros.

Note that the PMT and FV must have the same sign, since both are cash flows paid to the investor (paid by the bond issuer). The calculated PV will have the opposite sign from PMT and FV.

Future Value of an Annuity Due

Sometimes it is necessary to find the *FV of an annuity due* (FVA_D), an annuity where the annuity payments (or deposits) occur at the beginning of each compounding period. Fortunately, our financial calculators can be used to do this, but with one slight modification—the calculator must be set to the beginning-of-period (BGN) mode. To switch between the BGN and END modes on the TI, press [2nd] [BGN] [2nd] [SET]. When this is done, “BGN” will appear in the upper right corner of the display window. If the display indicates the desired mode, press [2nd] [QUIT]. You will normally want your calculator to be in the ordinary annuity (END) mode, so remember to switch out of BGN mode after working annuity due problems. Note that nothing appears in the upper right corner of the display window when the TI is set to the END mode. While annuity due payments are made or received at the beginning of each period, the FV of an annuity due is calculated as of the end of the last period.

Another way to compute the FV of an annuity due is to calculate the FV of an ordinary annuity, and simply multiply the resulting FV by $[1 + \text{periodic compounding rate } (I/Y)]$. Symbolically, this can be expressed as:

$$FVA_D = FVA_O \times (1 + I/Y)$$

The following examples illustrate how to compute the FV of an annuity due:

EXAMPLE: FV of an annuity due

What is the future value of an annuity that pays \$200 per year at the beginning of each of the next three years, commencing today, if the cash flows can be invested at an annual rate of 10%?

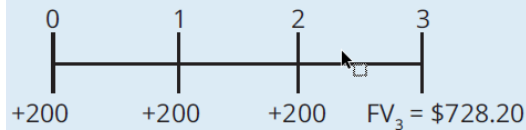
Answer:

Note in the time line in the following figure that the FV is computed as of the end of the last year in the life of the annuity, Year 3, even though the final payment occurs at the beginning of Year 3 (end of Year 2).

To solve this problem, put your calculator in the BGN mode ([2nd] [BGN] [2nd] [SET] [2nd] [QUIT] on the TI or [g] [BEG] on the HP), then input the relevant data and compute FV.

$$N = 3; I/Y = 10; PMT = -200; CPT \rightarrow FV = \$728.20$$

FV of an Annuity Due



Alternatively, we could calculate the FV for an ordinary annuity and multiply it by $(1 + I/Y)$. Leaving your calculator in the END mode, enter the following inputs:

$$N = 3; I/Y = 10; PMT = -200; CPT \rightarrow FVA_O = \$662.00$$

$$FVA_D = FVA_O \times (1 + I/Y) = 662 \times 1.10 = \$728.20$$

Present Value of an Annuity Due

While less common than those for ordinary annuities, some problems may require you to find the *PV of an annuity due* (PVA_D). Using a financial calculator, this really shouldn't be much of a problem. With an annuity due, *there is one less discounting period* since the first cash flow occurs at $t = 0$ and thus is already its PV. This implies that, all else equal, the PV of an annuity due will be greater than the PV of an ordinary annuity.

As you will see in the next example, there are two ways to compute the PV of an annuity due. The first is to put the calculator in the BGN mode and then input all the relevant variables (PMT, I/Y, and N) as you normally would. The second, and far easier way, is to treat the cash flow stream as an ordinary annuity over N compounding periods, and simply multiply the resulting PV by $[1 + \text{periodic compounding rate } (I/Y)]$.

Symbolically, this can be stated as:

$$PVA_D = PVA_O \times (1 + I/Y)$$

The advantage of this second method is that you leave your calculator in the END mode and will not run the risk of forgetting to reset it. Regardless of the procedure used, the computed PV is given as of the beginning of the first period, $t = 0$.

EXAMPLE: PV of an annuity due

Given a discount rate of 10%, what is the present value of an annuity that makes \$200 payments at the beginning of each of the next three years, starting today?

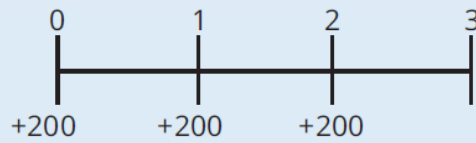
Answer:

First, let's solve this problem using the calculator's BGN mode. Set your calculator to the BGN mode ([2nd] [BGN] [2nd] [SET] [2nd] [QUIT] on the TI or [g] [BEG] on the HP), enter the relevant data, and compute PV.

$$N = 3; I/Y = 10; PMT = -200; CPT \rightarrow PVA_D = \$547.11$$

The time line for this problem is shown in the following figure:

PV of an Annuity Due



$$PV_0 = \$547.11$$

Alternatively, this problem can be solved by leaving your calculator in the END mode. First, compute the PV of an ordinary 3-year annuity. Then multiply this PV by $(1 + I/Y)$. To use this approach, enter the relevant inputs and compute PV.

$$N = 3; I/Y = 10; PMT = -200; CPT \rightarrow PVA_O = \$497.37$$

$$PVA_D = PVA_O \times (1 + I/Y) = \$497.37 \times 1.10 = \$547.11$$

Present Value of a Perpetuity

A **perpetuity** is a financial instrument that pays a fixed amount of money at set intervals over an *infinite* period of time. In essence, a perpetuity is a perpetual annuity. Most preferred stocks are examples of perpetuities since they promise fixed interest or dividend payments forever. Without going into all the excruciating mathematical details, the discount factor for a perpetuity is just one divided by the appropriate rate of return (i.e., $1/r$). Given this, we can compute the PV of a perpetuity.

$$PV_{\text{perpetuity}} = \frac{PMT}{\frac{I}{Y}}$$

The PV of a perpetuity is the fixed periodic cash flow divided by the appropriate periodic rate of return.

As with other TVM applications, you can solve for unknown variables in the $PV_{\text{perpetuity}}$ equation. In fact, you can solve for any one of the three relevant variables, given the values for the other two.

EXAMPLE: PV of a perpetuity

Kodon Corporation issues preferred stock that will pay \$4.50 per year in annual dividends beginning next year and plans to follow this dividend policy forever. Given an 8% rate of return, what is the value of Kodon's preferred stock today?

Answer:

Given that the value of the stock is the PV of all future dividends, we have:

$$PV_{\text{perpetuity}} = \frac{4.50}{0.08} = \$56.25$$

Thus, if an investor requires an 8% rate of return, the investor should be willing to pay \$56.25 for each share of Kodon's preferred stock. Note that the PV of a perpetuity is its value one period before its next payment.

EXAMPLE: PV of a deferred perpetuity

Assume the Kodon preferred stock in the preceding examples is scheduled to pay its first dividend in four years, and is non-cumulative (i.e., does not pay any dividends for the first three years). Given an 8% required rate of return, what is the value of Kodon's preferred stock today?

Answer:

As in the previous example, $PV_{\text{perpetuity}} = \frac{4.50}{0.08} = \56.25 , but because the first dividend is paid at $t = 4$, this PV is the value at $t = 3$. To get the value of the preferred stock today, we must discount this value for three periods: $\frac{56.25}{(1.08)^3} = \44.65 .

MODULE QUIZ 1.2

1. The amount an investor will have in 15 years if \$1,000 is invested today at an annual interest rate of 9% will be *closest* to:
 - A. \$1,350.
 - B. \$3,518.
 - C. \$3,642.
2. How much must be invested today, at 8% interest, to accumulate enough to retire a \$10,000 debt due seven years from today?
 - A. \$5,835.
 - B. \$6,123.
 - C. \$8,794.
3. An investor has just won the lottery and will receive \$50,000 per year at the end of each of the next 20 years. At a 10% interest rate, the present value of the winnings is *closest* to:
 - A. \$425,678.
 - B. \$637,241.
 - C. \$2,863,750.
4. An investor is to receive a 15-year, \$8,000 annuity, with the first payment to be received today. At an 11% discount rate, this annuity's worth today is *closest* to:
 - A. \$55,855.
 - B. \$57,527.
 - C. \$63,855.

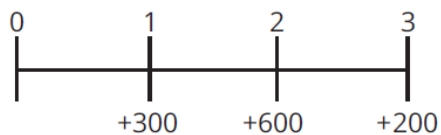
5. If \$1,000 is invested today and \$1,000 is invested at the beginning of each of the next three years at 12% interest (compounded annually), the amount an investor will have at the end of the fourth year will be *closest* to:
 - A. \$4,779.
 - B. \$5,353.
 - C. \$6,792.

6. Terry Corporation preferred stocks are expected to pay a \$9 annual dividend forever. If the required rate of return on equivalent investments is 11%, a share of Terry preferred should be worth:
 - A. \$81.82.
 - B. \$99.00.
 - C. \$122.22.

MODULE 1.3: UNEVEN CASH FLOWS

Many applications in investments and corporate finance evaluate a cash flow stream that is not equal from period to period. The time line in Figure 1.2 depicts such a cash flow stream.

Figure 1.2: Time Line for Uneven Cash Flows



This three-year cash flow series is not an annuity since the cash flows are different every year. In essence, this series of uneven cash flows is nothing more than a stream of annual single sum cash flows. Thus, to find the PV or FV of this cash flow stream, all we need to do is sum the PVs or FVs of the individual cash flows.

EXAMPLE: Computing the FV of an uneven cash flow series

Using a rate of return of 10%, compute the future value of the 3-year uneven cash flow stream described above at the end of the third year.

Answer:

The FV for the cash flow stream is determined by first computing the FV of each individual cash flow, then summing the FVs of the individual cash flows.

$$FV_1: PV = -300; I/Y = 10; N = 2; CPT \rightarrow FV = FV_1 = 363$$

$$FV_2: PV = -600; I/Y = 10; N = 1; CPT \rightarrow FV = FV_2 = 660$$

$$FV_3: PV = -200; I/Y = 10; N = 0; CPT \rightarrow FV = FV_3 = 200$$

$$\mathbf{FV \text{ of cash flow stream} = \sum FV_{\text{individual}} = 1,223}$$

EXAMPLE: Computing PV of an uneven cash flow series

Compute the present value of this 3-year uneven cash flow stream described previously using a 10% rate of return.

Answer:

This problem is solved by first computing the PV of each individual cash flow, then summing the PVs of the individual cash flows, which yields the PV of the cash flow stream. Again the signs of the cash flows are preserved.

$$PV_1: FV = 300; I/Y = 10; N = 1; CPT \rightarrow PV = PV_1 = -272.73$$

$$PV_2: FV = 600; I/Y = 10; N = 2; CPT \rightarrow PV = PV_2 = -495.87$$

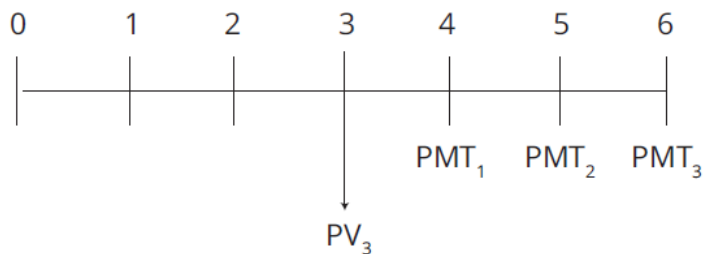
$$PV_3: FV = 200; I/Y = 10; N = 3; CPT \rightarrow PV = PV_3 = -150.26$$

$$PV \text{ of cash flow stream} = \sum PV_{\text{individual}} = \$918.86$$

LOS 1.d: Demonstrate the use of a time line in modeling and solving time value of money problems.

In most of the PV problems we have discussed, cash flows were discounted back to the current period. In this case, the PV is said to be indexed to $t = 0$, or the time index is $t = 0$. For example, the PV of a 3-year ordinary annuity that is indexed to $t = 0$ is computed at the beginning of Year 1 ($t = 0$). Contrast this situation with another 3-year ordinary annuity that does not start until Year 4 and extends to Year 6. You may also want to know the PV of this annuity at the beginning of Year 4, in which case the time index is $t = 3$. The time line for this annuity is presented in Figure 1.3.

Figure 1.3: Indexing Time Line to Other Than $t = 0$



The following examples will illustrate how to compute I/Y , N , or PMT in annuity problems:

EXAMPLE: Computing an annuity payment needed to achieve a given FV

At an expected rate of return of 7%, how much must be deposited at the end of each year for the next 15 years to accumulate \$3,000?

Answer:

To solve this problem, enter the three relevant known values and compute PMT .

$$N = 15; I/Y = 7; FV = +\$3,000; CPT \rightarrow PMT = -\$119.38 \text{ (ignore sign)}$$

EXAMPLE: Computing a loan payment

Suppose you are considering applying for a \$2,000 loan that will be repaid with equal end-of-year payments over the next 13 years. If the annual interest rate for the loan is 6%, how much will your payments be?

Answer:

The size of the end-of-year loan payment can be determined by inputting values for the three known variables and computing PMT.

$$N = 13; I/Y = 6; PV = -2,000; CPT \rightarrow PMT = \$225.92$$

EXAMPLE: Computing the number of periods in an annuity

How many \$100 end-of-year payments are required to accumulate \$920 if the discount rate is 9%?

Answer:

The number of payments necessary can be determined by inputting the relevant data and computing N.

$$I/Y = 9\%; FV = \$920; PMT = -\$100; CP \rightarrow TN = 7 \text{ years}$$

It will take seven annual \$100 payments, compounded at 9% annually, to accrue an investment value of \$920.

Professor's Note

Remember the sign convention! PMT and FV must have opposite signs or your calculator will issue an error message.

EXAMPLE: Computing the number of years in an ordinary annuity

Suppose you have a \$1,000 ordinary annuity earning an 8% return. How many annual end-of-year \$150 withdrawals can be made?

Answer:

The number of years in the annuity can be determined by entering the three relevant variables and computing N.

$$I/Y = 8; PMT = 150; PV = -1,000; CP \rightarrow TN = 9.9 \text{ years}$$

EXAMPLE: Computing the rate of return for an annuity

Suppose you have the opportunity to invest \$100 at the end of each of the next five years in exchange for \$600 at the end of the fifth year. What is the annual rate of return on this investment?

Answer:

The rate of return on this investment can be determined by entering the relevant data and solving for I/Y.

$$N = 5; FV = \$600; PMT = -100; CPT \rightarrow I/Y = 9.13\%$$

EXAMPLE: Computing the discount rate for an annuity

What rate of return will you earn on an ordinary annuity that requires a \$700 deposit today and promises to pay \$100 per year at the end of each of the next 10 years?

Answer:

The discount rate on this annuity is determined by entering the three known values and computing I/Y.

$$N = 10; PV = -700; PMT = 100; CPT \rightarrow I/Y = 7.07\%$$

Funding a Future Obligation

Many TVM applications will require you to determine the size of the deposit(s) that must be made over a specified period in order to meet a future liability, such as setting up a funding program for future college tuition or a retirement program. In most of these applications, the objective is to determine the size of the payment(s) or deposit(s) necessary to meet a particular monetary goal.

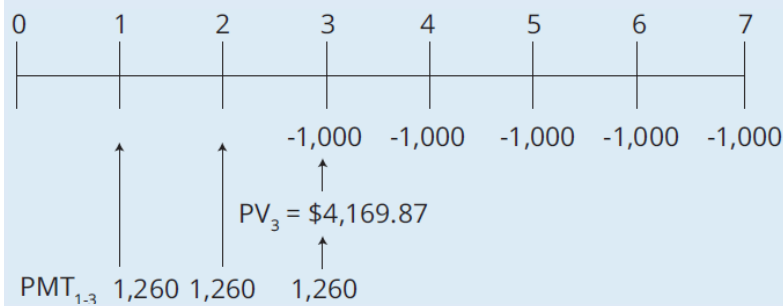
EXAMPLE: Computing the required payment to fund an annuity due

Suppose you must make five annual \$1,000 payments, the first one starting at the beginning of Year 4 (end of Year 3). To accumulate the money to make these payments, you want to make three equal payments into an investment account, the first to be made one year from today. Assuming a 10% rate of return, what is the amount of these three payments?

Answer:

The time line for this annuity problem is shown in the following figure:

Figure 1.4: Funding an Annuity Due



The first step in this type of problem is to determine the amount of money that must be available at the end of the third year in order to satisfy the payment requirements. This amount is the PV of a 5-year annuity due at the beginning of Year 4 (end of Year 3). To determine this amount, set your calculator to the BGN mode, enter the relevant data, and compute PV.

$$N = 5; I/Y = 10; PMT = -1,000; CPT \rightarrow PV = PV_3 = \$4,169.87$$

PV_3 becomes the FV that you need three years from today from your three equal end-of-year deposits. To determine the amount of the three payments necessary to meet this funding requirement, be sure that your calculator is in the END mode, input the relevant data, and compute PMT.

$$N = 3; I/Y = 10; FV = -4,169.87; CPT \rightarrow PMT = \$1,259.78$$

The second part of this problem is an ordinary annuity. If you changed your calculator to BGN mode and failed to put it back in the END mode, you will get a PMT of \$1,145, which is incorrect.

The Connection Between Present Values, Future Values, and Series of Cash Flows

As we have explained in the discussion of annuities and series of uneven cash flows, the sum of the present values of the cash flows is the present value of the series. The sum of the future values (at some future time = n) of a series of cash flows is the future value of that series of cash flows.

One interpretation of the present value of a series of cash flows is how much would have to be put in the bank today in order to make these future withdrawals and exhaust the account with the final withdrawal. Let's illustrate this with cash flows of \$100 in Year 1, \$200 in Year 2, \$300 in Year 3, and an assumed interest rate of 10%.

Calculate the present value of these three cash flows as:

$$\frac{100}{1.1} + \frac{200}{1.1^2} + \frac{300}{1.1^3} = \$481.59$$

If we put \$481.59 in an account yielding 10%, at the end of the year we would have $481.59 \times 1.1 = \$529.75$. Withdrawing \$100 would leave \$429.75.

Over the second year, the \$429.75 would grow to $429.75 \times 1.1 = \$472.73$. Withdrawing \$200 would leave \$272.73.

Over the third year, \$272.73 would grow to $272.73 \times 1.1 = \$300$, so that the last withdrawal of \$300 would empty the account.

The interpretation of the future value of a series of cash flows is straightforward as well. The FV answers the question, "How much would be in an account when the last of a series of deposits is made?" Using the same three cash flows—\$100, \$200, and \$300—and the same interest rate of 10%, we can calculate the future value of the series as:

$$100(1.1)^2 + 200(1.1) + 300 = \$641$$

This is simply the sum of the $t = 3$ value of each of the cash flows. Note that the $t = 3$ value and the $t = 0$ (present) value of the series are related by the interest rate, $481.59(1.1)^3 = 641$.

The \$100 cash flow (deposit) comes at $t = 1$, so it will earn interest of 10% compounded for two periods (until $t = 3$). The \$200 cash flow (deposit) will earn 10% between $t = 2$ and $t = 3$, and the final cash flow (deposit) of \$300 is made at $t = 3$, so \$300 is the future ($t = 3$) value of that cash flow.

We can also look at the future value in terms of how the account grows over time. At $t = 1$ we deposit \$100, so at $t = 2$ it has grown to \$110 and the \$200 deposit at $t = 2$ makes the account balance \$310. Over the next period, the \$310 grows to $310 \times 1.1 = \$341$ at $t = 3$, and the addition of the final \$300 deposit puts the account balance at \$641. This is, of course, the future value we calculated initially.

Professor's Note

This last view of the future value of a series of cash flows suggests a quick way to calculate the future value of an uneven cash flow series. The process described previously for the future value of a series of end-of-period payments can be written mathematically as $[(100 \times 1.1) + 200] \times 1.1 + 300 = 641$, and this might be a quick way to do some future value problems. On your TI calculator, you would enter $100 \times 1.1 + 200 = \times 1.1 + 300 =$ to get 641.

Note that questions on the future value of an *annuity due* refer to the amount in the account one period after the last deposit is made. If the three deposits considered here were made at the beginning of each period (at $t = 0, 1, 2$) the amount in the account at the end of three years ($t = 3$) would be 10% higher (i.e., $641 \times 1.1 = \$705.10$).

Professor's Note

The cash flow additivity principle is also covered in the main reading and, therefore, is examinable.

The **cash flow additivity principle** refers to the fact that present value of any stream of cash flows equals the sum of the present values of the cash flows. There are different applications of this principle in time value of money problems. If we have two series of cash flows, the sum of the present values of the two series is the same as the present values of the two series taken together, adding cash flows that will be paid at the same point in time. We can also divide up a series of cash flows any way we like, and the present value of the "pieces" will equal the present value of the original series.

EXAMPLE: Additivity principle

A security will make the following payments at the end of the next four years: \$100, \$100, \$400, and \$100. Calculate the present value of these cash flows using the concept of the present value of an annuity when the appropriate discount rate is 10%.

Answer:

We can divide the cash flows so that we have:

$t = 1$	$t = 2$	$t = 3$	$t = 4$	
100	100	100	100	cash flow series #1
0	0	300	0	cash flow series #2
\$100	\$100	\$400	\$100	

The additivity principle tells us that to get the present value of the original series, we can just add the present values of series #1 (a 4-period annuity) and series #2 (a single payment three periods from now).

For the annuity: $N = 4$; $PMT = 100$; $FV = 0$; $I/Y = 10$; $CPT \rightarrow PV = -\$316.99$

For the single payment: $N = 3$; $PMT = 0$; $FV = 300$; $I/Y = 10$; $CPT \rightarrow PV = -\$225.39$

The sum of these two values is $316.99 + 225.39 = \$542.38$.

The sum of these two (present) values is identical (except for rounding) to the sum of the present values of the payments of the original series:

$$\frac{100}{1.1} + \frac{100}{1.1^2} + \frac{400}{1.1^3} + \frac{100}{1.1^4} = \$542.38$$

MODULE QUIZ 1.3

1. An analyst estimates that XYZ's earnings will grow from \$3.00 a share to \$4.50 per share over the next eight years. The rate of growth in XYZ's earnings is *closest* to:
 - A. 4.9%.
 - B. 5.2%.
 - C. 6.7%.
2. If \$5,000 is invested in a fund offering a rate of return of 12% per year, approximately how many years will it take for the investment to reach \$10,000?
 - A. 4 years.
 - B. 5 years.
 - C. 6 years.
3. An investment is expected to produce the cash flows of \$500, \$200, and \$800 at the end of the next three years. If the required rate of return is 12%, the present value of this investment is *closest* to:
 - A. \$835.
 - B. \$1,175.
 - C. \$1,235.
4. If \$10,000 is invested today in an account that earns interest at a rate of 9.5%, what is the value of the equal withdrawals that can be taken out of the account at the end of each of the next five years if the investor plans to deplete the account at the end of the time period?
 - A. \$2,453.
 - B. \$2,604.
 - C. \$2,750.
5. Given an 11% rate of return, the amount that must be put into an investment account at the end of each of the next 10 years in order to accumulate \$60,000 to pay for a child's education is *closest* to:
 - A. \$2,500.
 - B. \$3,588.
 - C. \$4,432.

6. An investor will receive an annuity of \$4,000 a year for 10 years. The first payment is to be received five years from today. At a 9% discount rate, this annuity's worth today is *closest* to:
- A. \$16,684.
 - B. \$18,186.
 - C. \$25,671.

MODULE 1.4: COMPOUNDING FREQUENCIES

LOS 1.e: Calculate the solution for time value of money problems with different frequencies of compounding.

While the conceptual foundations of TVM calculations are not affected by the compounding period, more frequent compounding does have an impact on FV and PV computations. Specifically, since an increase in the frequency of compounding increases the effective rate of interest, it also *increases* the FV of a given cash flow and *decreases* the PV of a given cash flow.

EXAMPLE: The effect of compounding frequency on FV and PV

Compute the FV one year from now of \$1,000 today and the PV of \$1,000 to be received one year from now using a stated annual interest rate of 6% with a range of compounding periods.

Answer:

Compounding Frequency Effect

Compounding Frequency	Interest Rate per Period	Effective Annual Rate	Future Value	Present Value
Annual ($m = 1$)	6.000%	6.00%	\$1,060.00	\$943.396
Semiannual ($m = 2$)	3.000	6.090	1,060.90	942.596
Quarterly ($m = 4$)	1.500	6.136	1,061.36	942.184
Monthly ($m = 12$)	0.500	6.168	1,061.68	941.905
Daily ($m = 365$)	0.016438	6.183	1,061.83	941.769

There are two ways to use your financial calculator to compute PVs and FVs under different compounding frequencies:

1. Adjust the number of periods per year (P/Y) mode on your calculator to correspond to the compounding frequency (e.g., for quarterly, $P/Y = 4$). **WE DO NOT RECOMMEND THIS APPROACH!**
2. Keep the calculator in the annual compounding mode ($P/Y = 1$) and enter I/Y as the interest rate per compounding period, and N as the number of compounding periods in the investment horizon. Letting m equal the number of compounding periods per year, the basic formulas for the calculator input data are determined as follows:
 - $I/Y = \text{the annual interest rate} / m$
 - $N = \text{the number of years} \times m$

The computations for the FV and PV amounts in the previous example are:

- PV_A: FV = -1,000; I/Y = 6 / 1 = 6; N = 1 × 1 = 1:
CPT → PV = PV_A = 943.396
- PV_S: FV = -1,000; I/Y = 6 / 2 = 3; N = 1 × 2 = 2:
CPT → PV = PV_S = 942.596
- PV_Q: FV = -1,000; I/Y = 6 / 4 = 1.5; N = 1 × 4 = 4:
CPT → PV = PV_Q = 942.184
- PV_M: FV = -1,000; I/Y = 6 / 12 = 0.5; N = 1 × 12 = 12:
CPT → PV = PV_M = 941.905
- PV_D: FV = -1,000; I/Y = 6 / 365 = 0.016438; N = 1 × 365 = 365:
CPT → PV = PV_D = 941.769
- FV_A: PV = -1,000; I/Y = 6 / 1 = 6; N = 1 × 1 = 1:
CPT → FV = FV_A = 1,060.00
- FV_S: PV = -1,000; I/Y = 6 / 2 = 3; N = 1 × 2 = 2:
CPT → FV = FV_S = 1,060.90
- FV_Q: PV = -1,000; I/Y = 6 / 4 = 1.5; N = 1 × 4 = 4:
CPT → FV = FV_Q = 1,061.36
- FV_M: PV = -1,000; I/Y = 6 / 12 = 0.5; N = 1 × 12 = 12:
CPT → FV = FV_M = 1,061.68
- FV_D: PV = -1,000; I/Y = 6 / 365 = 0.016438; N = 1 × 365 = 365:
CPT → FV = FV_D = 1,061.83

EXAMPLE: FV of a single sum using quarterly compounding

Compute the FV of \$2,000 today, five years from today using an interest rate of 12%, compounded quarterly.

Answer:

To solve this problem, enter the relevant data and compute FV:

$$N = 5 \times 4 = 20; I/Y = 12 / 4 = 3; PV = -\$2,000; CPT \rightarrow FV = \$3,612.22$$

EXAMPLE: Growth with quarterly compounding

John plans to invest \$2,500 in an account that will earn 8% per year with quarterly compounding. How much will be in the account at the end of two years?

Answer:

There are eight quarterly compounding periods in two years, and the effective quarterly rate is $8 / 4 = 2\%$. The account will grow to $2,500(1.02)^8 = \$2,929.15$. Alternatively, since the EAR is $1.02^4 - 1 = 0.082432$, we can grow the \$2,500 at 8.2432% for two years to get $2,500(1.082432)^2 = \$2,929.15$, which is the same result.

EXAMPLE: Present value with monthly compounding

Alice would like to have \$5,000 saved in an account at the end of three years. If the return on the account is 9% per year with monthly compounding, how much must Alice deposit today in order to reach her savings goal in three years?

Answer:

The effective monthly rate is $9 / 12 = 0.75\%$, and we can calculate the present value of \$5,000 three years (36 months) from now as $5,000 / (1.0075)^{36} = \$3,820.74$. Alternatively, since the EAR is $1.0075^{12} - 1 = 0.093807$, we can calculate the present value by discounting 5,000 at the EAR for three years. $5,000 / 1.093807^3 = \$3,820.74$, which is the same result.

LOS 1.f: Calculate and interpret the effective annual rate, given the stated annual interest rate and the frequency of compounding.

Financial institutions usually quote rates as stated annual interest rates, along with a compounding frequency, as opposed to quoting rates as periodic rates—the rate of interest earned over a single compounding period. For example, a bank will quote a savings rate as 8%, compounded quarterly, rather than 2% per quarter. The rate of interest that investors actually realize as a result of compounding is known as the **effective annual rate (EAR)** or **effective annual yield (EAY)**. EAR represents the annual rate of return actually being earned *after adjustments have been made for different compounding periods*.

EAR may be determined as follows:

$$\text{EAR} = (1 + \text{periodic rate})^m - 1$$

where:

periodic rate = stated annual rate/m

m = the number of compounding periods per year

Obviously, the EAR for a stated rate of 8% *compounded annually* is not the same as the EAR for 8% *compounded semiannually*, or *quarterly*. Indeed, whenever compound interest is being used, the stated rate and the actual (effective) rate of interest are equal only when interest is compounded annually. Otherwise, the greater the compounding frequency, the greater the EAR will be in comparison to the stated rate.

The computation of EAR is necessary when comparing investments that have different compounding periods. This allows for an apples-to-apples rate comparison.

EXAMPLE: Computing ear

Compute EAR if the stated annual rate is 12%, compounded quarterly.

Answer:

Here $m = 4$, so the periodic rate is $\frac{12}{4} = 3\%$.

Thus, $\text{EAR} = (1 + 0.03)^4 - 1 = 1.1255 - 1 = 0.1255 = 12.55\%$.

This solution uses the $[y^x]$ key on your financial calculator. The exact keystrokes on the TI for the above computation are 1.03 $[y^x]$ 4 [=]. On the HP, the strokes are 1.03 [ENTER] 4 $[y^x]$.

EXAMPLE: Computing ears for a range of compounding frequencies

Using a stated rate of 6%, compute EARs for semiannual, quarterly, monthly, and daily compounding.

Answer:

EAR with:

semiannual compounding	$= (1 + 0.03)^2 - 1$	$= 1.06090 - 1 = 0.06090 = 6.090\%$
quarterly compounding	$= (1 + 0.015)^4 - 1$	$= 1.06136 - 1 = 0.06136 = 6.136\%$
monthly compounding	$= (1 + 0.005)^{12} - 1$	$= 1.06168 - 1 = 0.06168 = 6.168\%$
daily compounding	$= (1 + 0.00016438)^{365} - 1$	$= 1.06183 - 1 = 0.06183 = 6.183\%$

Notice here that the EAR increases as the compounding frequency increases.

Professor's Note

The limit of shorter and shorter compounding periods is called continuous compounding, which we will address in a later reading.

MODULE QUIZ 1.4

1. What is the effective annual rate for a credit card that charges 18% compounded monthly?
 - A. 15.38%.
 - B. 18.81%.
 - C. 19.56%.
2. Given daily compounding, the growth of \$5,000 invested for one year at 12% interest will be *closest* to:
 - A. \$5,600.
 - B. \$5,628.
 - C. \$5,637.
3. An investor is looking at a \$150,000 home. If 20% must be put down and the balance is financed at a stated annual rate of 9% over the next 30 years, what is the monthly mortgage payment?
 - A. \$799.33.
 - B. \$895.21.
 - C. \$965.55.

Key Concepts

LOS 1.a

An interest rate can be interpreted as the rate of return required in equilibrium for a particular investment, the discount rate for calculating the present value of future cash flows, or as the opportunity cost of consuming now, rather than saving and investing.

LOS 1.b

The real risk-free rate is a theoretical rate on a single-period loan when there is no expectation of inflation. Nominal risk-free rate = real risk-free rate + expected inflation rate.

Securities may have several risks, and each increases the required rate of return. These include default risk, liquidity risk, and maturity risk.

The required rate of return on a security = real risk-free rate + expected inflation + default risk premium + liquidity premium + maturity risk premium.

LOS 1.c

Future value: $FV = PV(1 + I/Y)^N$

Present value: $PV = FV / (1 + I/Y)^N$

An annuity is a series of equal cash flows that occurs at evenly spaced intervals over time. Ordinary annuity cash flows occur at the end of each time period. Annuity due cash flows occur at the beginning of each time period.

Perpetuities are annuities with infinite lives (perpetual annuities):

$$PV_{\text{perpetuity}} = \frac{PMT}{I/Y}$$

The present (future) value of any series of cash flows is equal to the sum of the present (future) values of the individual cash flows.

LOS 1.d

Constructing a time line showing future cash flows will help in solving many types of TVM problems. Cash flows occur at the end of the period depicted on the time line. The end of one period is the same as the beginning of the next period. For example, a cash flow at the beginning of Year 3 appears at time $t = 2$ on the time line.

LOS 1.e

For non-annual time value of money problems, divide the stated annual interest rate by the number of compounding periods per year, m , and multiply the number of years by the number of compounding periods per year.

LOS 1.f

The effective annual rate when there are m compounding periods =

$\left(1 + \frac{\text{stated annual rate}}{m}\right)^m - 1$. Each dollar invested will grow to

$\left(1 + \frac{\text{stated annual rate}}{m}\right)^m$ in one year.

Module Quiz Answers

Module Quiz 1.1

1. **c** Interest rates can be interpreted as required rates of return, discount rates, or opportunity costs of current consumption. A risk premium can be, but is not always, a component of an interest rate. (Module 1.1, LOS 1.a, 1.b)
2. **a** Real interest rates are those that have been adjusted for inflation. (Module 1.1, LOS 1.b)

Module Quiz 1.2

1. **c** $N = 15$; $I/Y = 9$; $PV = -1,000$; $PMT = 0$; $CPT \rightarrow FV = \$3,642.48$
(Module 1.2, LOS 1.c)
2. **a** $N = 7$; $I/Y = 8$; $FV = -10,000$; $PMT = 0$; $CPT \rightarrow PV = \$5,834.90$
(Module 1.2, LOS 1.c)
3. **a** $N = 20$; $I/Y = 10$; $PMT = -50,000$; $FV = 0$; $CPT \rightarrow PV = \$425,678.19$
(Module 1.2, LOS 1.c)
4. **c** This is an annuity due. Switch to BGN mode: $N = 15$; $PMT = -8,000$; $I/Y = 11$; $FV = 0$; $CPT \rightarrow PV = 63,854.92$. Switch back to END mode. (Module 1.2, LOS 1.c)
5. **b** The key to this problem is to recognize that it is a 4-year annuity due, so switch to BGN mode: $N = 4$; $PMT = -1,000$; $PV = 0$; $I/Y = 12$; $CPT \rightarrow FV = 5,352.84$. Switch back to END mode.
(Module 1.2, LOS 1.c)
6. **a** $9 / 0.11 = \$81.82$
(Module 1.2, LOS 1.c)

Module Quiz 1.3

1. **b** $N = 8$; $PV = -3$; $FV = 4.50$; $PMT = 0$; $CPT \rightarrow I/Y = 5.1989$
(Module 1.3, LOS 1.d)
2. **c** $PV = -5,000$; $I/Y = 12$; $FV = 10,000$; $PMT = 0$; $CPT \rightarrow N = 6.12$

Note to HP 12C users: One known problem with the HP 12C is that it does not have the capability to round. In this particular question, you will come up with 7, although the correct answer is 6.1163. CFA Institute is aware of this problem, and hopefully you will not be faced with a situation on exam day where the incorrect solution from the HP is one of the answer choices. (Module 1.3, LOS 1.d)

3. **b** Add up the present values of each single cash flow.
 $PV_1 = N = 1$; $FV = -500$; $I/Y = 12$; $CPT \rightarrow PV = 446.43$

$PV_2 = N = 2; FV = -200; I/Y = 12; CPT \rightarrow PV = 159.44$

$PV_3 = N = 3; FV = -800; I/Y = 12; CPT \rightarrow PV = 569.42$

Hence, $446.43 + 159.44 + 569.42 = \$1,175.29$.

(Module 1.3, LOS 1.d)

4. **b** $PV = -10,000; I/Y = 9.5; N = 5; FV = 0; CPT \rightarrow PMT = \$2,604.36$

(Module 1.3, LOS 1.d)

5. **b** $N = 10; I/Y = 11; FV = -60,000; PV = 0; CPT \rightarrow PMT = \$3,588.08$

(Module 1.3, LOS 1.d)

6. **b** Two steps: (1) Find the PV of the 10-year annuity: $N = 10; I/Y = 9; PMT = -4,000; FV = 0; CPT \rightarrow PV = 25,670.63$. This is the present value as of the end of Year 4; (2) Discount PV of the annuity back four years: $N = 4; PMT = 0; FV = -25,670.63; I/Y = 9; CPT \rightarrow PV = 18,185.72$.

(Module 1.3, LOS 1.d)

Module Quiz 1.4

1. **c** $EAR = [(1 + (0.18 / 12))]^{12} - 1 = 19.56\%$

(Module 1.4, LOS 1.f)

2. **c** $N = 1 \times 365 = 365; I/Y = 12 / 365 = 0.0328767; PMT = 0; PV = -5,000; CPT \rightarrow FV = \$5,637.37$

(Module 1.4, LOS 1.e)

3. **c** $N = 30 \times 12 = 360; I/Y = 9 / 12 = 0.75; PV = -150,000(1 - 0.2) = -120,000; FV = 0; CPT \rightarrow PMT = \965.55

(Module 1.4, LOS 1.e)