LM02 Valuation of Contingent Claims

Version 1.0

1. Introduction

A contingent claim is a derivative instrument that gives its owner a right but not an obligation to a payoff determined by an underlying asset, rate, or another derivative.

This reading covers:

- Principles of the no-arbitrage approach which forms the basis of option valuation models
- Option valuation using the binomial model based on discrete time
- Option valuation using the Black-Scholes-Merton (BSM) model based on continuous time
- Black model applied to futures options, interest rate options, and swaptions
- Option Greeks and implied volatility

1.1 Principles of a No-Arbitrage Approach to Valuation

The no-arbitrage approach is built on the **law of one price**. According to this law, if two investments have the same future cash flows, then they should have the same price. The principle of no arbitrage means that the correct price or value of an option is one which ensures no arbitrage profits.

To understand option valuation models, we have to think like an arbitrageur. An arbitrageur seeks mispricing between the option price and the underlying spot price to make profits. In doing so, the arbitrageur follows two rules:

- Rule #1: Do not use your own money – the arbitrageur borrows money to purchase the underlying and invests proceeds from short selling transactions at the risk-free rate.
- Rule #2: Do not take any price risk – the arbitrageur eliminates any market price risk related to the underlying and the derivatives used.

Option valuation is based on the following assumptions:

- Replicating instruments are identifiable and investable.
- There are no market frictions, such as transaction costs and taxes.
- Short selling is allowed with full use of proceeds.
- The underlying instrument follows a known statistical distribution.
- Borrowing and lending at a risk-free interest rate is available.

2. Binomial Option Valuation Model

The binomial option valuation model is based on the no-arbitrage approach. This model is typically used to value path-dependent options, i.e., options whose value not only depend on the value of the underlying at expiration but also how it got there.

Instructor's Note:

The binomial option valuation model covered in this section can be used to value both path-dependent as well as path-independent options. The BSM model covered in the next section can be can be used to value path-independent options only.

European options are path-independent options because their values depend only on the value of the underlying at expiration. American options are path-dependent options because they can be exercised prior to expiration.

Following key symbols and terms are used in the valuation of options:

- S_t = price of the underlying at Time t, where t is expressed as a fraction of years.
- T = initial time to expiration, expressed as a fraction of years.
- S_0= price of the underlying at initiation of the contract.
- S_T = price of the underlying at expiration date.
- c_t = European-style call price at Time 't'
- C_t = American-style call price at Time 't'
- c = c_0 at initiation date, subscripts are removed
- p_t = European-style put price at Time 't'
- P_t = American-style put price at Time 't'
- p = p_0 at initiation date, subscripts are removed
- X = option exercise price (also called the strike price)

The exercise values for call and put options are given as:

- $c_T = Max(0, S_T – X)$
- $p_T = Max(0, X – S_T)$

A call option is in the money when the price of the underlying is higher than the exercise price. A put option is in the money when the price of the underlying is lower than the exercise price.

Example: On 15 November, a 90-day European-style call option with an exercise price of \$50.00 is priced at \$4.60. The underlying is trading at \$53.00. Hence, T = 90/365, S_0 = 53.00, X = 50.00, and c_0 = 4.60.

The option price has two components: 1) an exercise value component, also called intrinsic value, and 2) a time value component. Hence, we can say: c = exercise value + time value.

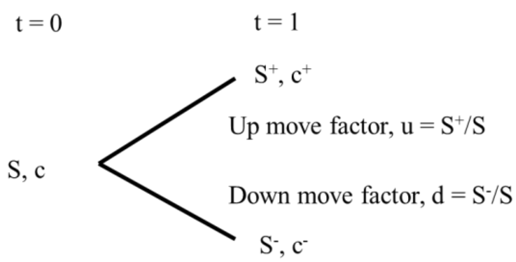
- For a call option, the exercise value component is S – X if S > X. In our example, the exercise value is 53.00 – 50.00 = 3.00.
- The time value component is equal to the option price minus that exercise value: 4.60 – 3.00 = 1.60.

The time value declines as time passes and becomes equal to zero at option expiration. At expiration, call and put values are equal to their intrinsic value or exercise value.

3. One-Period Binomial Model

No Arbitrage Approach (Call Options)

The one-period binomial model can be understood using the figure below:



At t = 0, a call option has value c and the underlying stock has value S. At t = 1, the stock can follow only two possible paths, it will either go up to S^+ or go down to S^-. The up factor, u, is equal to S^+/S and the down factor, d, is equal to S^-/S. If the stock goes to S^+, the call option value is c^+ and if the stock goes down to S^-, the call option value is c^-.

If a trader buys one call option, he faces price risk because the call option could go up or down in value. The investor can hedge this risk by shorting h units of the underlying stock.

Hence at t = 0, the trader's portfolio is –hS + c. We have a minus sign because h units of the stock are shorted. At t = 1, the portfolio's value is $hS^+ - c^+$ if the stock goes up and $hS^- - c^-$ if the stock goes down. Equating these two values ($hS^+ - c^+ = hS^- - c^-$) and solving for h gives:

$$h = \frac{c^+ - c^-}{S^+ - S^-}$$

This is called the hedge ratio. It makes the trader indifferent to the movement of the underlying. By creating a hedged portfolio, the trader has eliminated price risk and satisfied Rule 2: "Do not take any price risk". However, for the transaction to be completely arbitrage-free, the trader still needs to satisfy Rule 1: "Do not use your own money". This can happen if the trader borrows the present value (PV) of $-hS^- + c^-$.

For a no-arbitrage scenario, we have: $c - hS = PV(-hS^- + c^-)$ where PV stands for present value. Since $-hS^- + c^- = -hS^+ + c^+$, we can also say: $c - hS = PV(-hS^+ + c^+)$. Solving for c, we get the no-arbitrage single-period valuation equation for call options:

$c = hS + PV(-hS^- + c^-)$

or, $c = hS + PV(-hS^+ + c^+)$

Based on these expressions, it can be said that:

- A call option is equivalent to holding h units of the underlying and financing $-PV(-hS^- + c^-)$.
- A call option can be interpreted as a leveraged position in the underlying.

Example: Call Option Value

A non-dividend-paying stock is currently trading at $50.00. A call option has one year to mature, the periodically compounded risk-free interest rate is 7%, and the exercise price is $50.00. Assume a single-period binomial option valuation model, where u = 1.25 and d = 0.80. Estimate the option value.

Solution:
$S^+ = 50.00*1.25 = 62.50$, $c^+ = 12.50$
$S^- = 50.00*0.80 = 40.00$, $c^- = 0$

$$h = \frac{c^+ - c^-}{S^+ - S^-} = \frac{12.50 - 0}{62.50 - 40.00} = 0.56$$

$c = hS + PV(-hS^- + c^-) = 0.56*50 + PV(-0.56*40 + 0) = 0.56*50 + (-0.56*40 + 0)/1.07$
$= 28.00 - 20.93 = 7.07$.

Notice that buying a call option for $7.07 is equivalent to buying 0.56 units of the underlying stock for $28.00. This purchase is partially financed (20.93) such that the effective payment is $7.07.

No-Arbitrage Approach (Put Options)
For put options, the no-arbitrage single-period valuation equation is:

$p = hS + PV(-hS^- + p^-)$

or, equivalently, $p = hS + PV(-hS^+ + p^+)$, where the hedge ratio h is given by the expression:

$$h = \frac{p^+ - p^-}{S^+ - S^-}$$

Note that the hedge ratio is negative because p^+ is less than p^-.

Now, consider the equation: $p = hS + PV(-hS^- + p^-)$. The term hS is negative because h is negative. This implies short selling the underlying stock. The term $PV(-hS^- + p^-)$ is positive because h is negative. This implies lending $PV(-hS^- + p^-)$. Hence, it can be said that a put option is equivalent to shorting the underlying and lending $PV(-hS^+ + p^+)$.

Example: Put Option Value
A non-dividend-paying stock is currently trading at $50. A put option has one year to mature, the periodically compounded risk-free interest rate is 7%, and the exercise price is $50. Assume a single-period binomial option valuation model, where u = 1.25 and d = 0.80. Estimate the option value.

Solution:
$S^+ = 50.00*1.25 = 62.50$, $p^+ = 0$, $S^- = 50.00*0.80 = 40.00$, $p^- = 10$.
$h = (0 - 10)/(62.50 - 40.00) = -0.44$.
$p = hS + PV(-hS^+ + p^+) = -0.44*50 + PV(- -0.44*62.5 + 0) = -0.44*50 + (0.44*62.5 + 0)/1.07$
$= -22.00 + 25.70 = 3.70$.

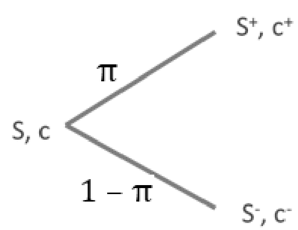
Notice that buying a put option for \$3.70 is equivalent to short selling 0.44 units of the underlying stock for \$22.00 and lending \$25.70.

Expectations Approach
Option values can also be calculated using the expectations approach. It will give us the same result as the no-arbitrage approach. According to the expectations approach, call and put values can be expressed:

$c = PV[\pi c^+ + (1 - \pi)c^-]$ and
$p = PV[\pi p^+ + (1 - \pi)p^-]$

where:
π = the risk-neutral probability of an up move = $(1 + r - d) / (u - d)$

$1 - \pi$ = the risk neutral probability of a down move



The expected terminal option payoffs can be expressed as:

$E(c_1) = \pi c^+ + (1 - \pi)c^-$ and
$E(p_1) = \pi p^+ + (1 - \pi)p^-$

where c_1 and p_1 = values of the options at Time 1. The discount rate for present value and future value is the risk-free rate, 'r'. Under the expectations approach option values can be written as:

$c = PV[E(c_1)]$ and
$p = PV[E(p_1)]$

With the expectations approach, the probability, π, is objectively determined and is called the risk-neutral (RN) probability. No assumption is made regarding the arbitrageur's risk preferences. The discount rate is not risk-adjusted. It is simply the risk-free interest rate.

Example: Call Value and Put Value using the Expectations Approach.
(*This is based on Example 3 of the curriculum.*)

A non-dividend-paying stock is currently trading at €100. A call option has one year to mature, the periodically compounded risk-free interest rate is 5.15%, and the exercise price is €100. Assume a single-period binomial option valuation model, where u = 1.35 and d = 0.74. What is the call option value and put option using the expectation approach?

Solution:

$S^+ = uS = 1.35(100) = 135$

$S^- = dS = 0.74(100) = 74$

$c^+ = \text{Max}(0, uS – X) = \text{Max}(0, 135 – 100) = 35$

$c^- = \text{Max}(0, dS – X) = \text{Max}(0, 74 – 100) = 0$

$p^+ = \text{Max}(0, 100 – uS) = \text{Max}(0, 100 – 135) = 0$

$p^- = \text{Max}(0, 100 – dS) = \text{Max}(0, 100 – 74) = 26$

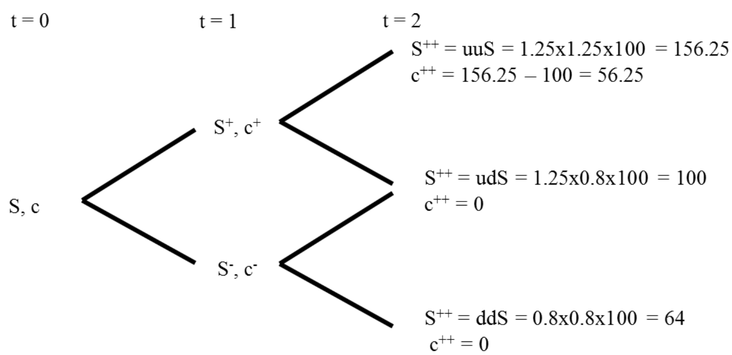
$\pi = [1 + r – d]/(u – d) = (1 + 0.0515 – 0.74)/(1.35 – 0.74) = 0.51$

$c = PV[\pi c^+ + (1 – \pi)c^-] = [(0.51)35 + (1 – 0.51)0] / 1.0515 = €17.00$

$p = PV[\pi p^+ + (1 – \pi)p^-] = [(0.51)0 + (1 – 0.51)26] / 1.0515 = €12.11$

4. Two-Period Binomial Model (Call Options)

The two-period binomial lattice can be perceived as three, one-period binomial lattices. Shown below is a two-period lattice based on an example in the curriculum.



The three possible values of the underlying at Time 2, are S^{++} (an up move that occurs twice), S^{--} (a down move that occurs twice), and $S^{+-} = S^{-+}$ (either an up move followed by a down move or a down move followed by an up move). We assume the up and down factors are constant such that the lattice recombines. Hence, $S^{+-} = S^{-+}$. For the lattice show above, we assume u = 1.25, d = 0.8, and S_0 = 100. Hence: $S^{+-} = 1.25(0.8)100 = 100$ and $S^{-+} = 0.8(1.25)100 = 100$. The middle node at Time 2 is 100 and can be reached from either of two paths: $S^{++} = 1.25 \times 1.25 \times 100 = 156.25$ and $S^{--} = 0.8 \times 0.8 \times 100 = 64$. If the stock goes up twice, the call option is in the money and $c^{++} = S – X = 156.25 – 100 = 56.25$.

The two-period binomial option valuation model can be used to demonstrate two important concepts, self-financing and dynamic replication.

- **Self-financing** means that the replicating portfolio will not require any additional funds and it is dynamically rebalanced. Additional funds, if required, are borrowed.
- **Dynamic replication** means the option payoffs can be replicated by following a planned trading strategy.

The option values under the two-period binomial model using the expectations approach are:

$c = PV[\pi^2 c^{++} + 2\pi(1 - \pi)c^{+-} + (1 - \pi)^2 c^{--}]$

$p = PV[\pi^2 p^{++} + 2\pi(1 - \pi)p^{+-} + (1 - \pi)^2 p^{--}]$

As indicated before, π, the risk-neutral probability of an up move, is (1 + r – d) / (u – d).

The value of a put option can also be determined using the put-call parity. The put-call parity relation asserts that a long stock and long put position is equivalent to lending the present value of the exercise price at the risk-free rate and a long call.

S + p = PV(X) + c and hence p = PV(X) + c – S

Example: Two Period Binomial Model Call Valuation
(*This is based on Example 5 of the curriculum.*)

You observe a €50 price for a non-dividend-paying stock. The call option and a put option on this stock have two years to mature, the periodically compounded risk-free interest rate is 5%, the exercise price is €50, u = 1.356, and d = 0.744. Assume the options are European-style.

1. What is risk-neutral up move probability?
2. What is the call option value?
3. What is the put option value?

Solution:

$\pi = (1 + r - d)/(u - d) = (1 + 0.05 - 0.744)/(1.356 - 0.744) = 0.50$ or 50%

$c^{++} = Max(0,uuS - X) = Max[0,1.356^2(50) - 50] = 41.9368$
$c^{-+} = c^{+-} = Max(0,udS - X) = Max[0,1.356(0.744)(50) - 50] = 0.44320$
$c^{--} = Max(0,ddS - X) = Max[0,0.744^2(50) - 50] = 0.0$

With this information, we can compute the call option value:

$c = PV[\pi^2 c^{++} + 2\pi(1 - \pi)c^{+-} + (1 - \pi)^2 c^{--}]$
$= [1/(1 + 0.05)]^2[0.5^2 (41.9368) + 2(0.5)(1 - 0.5)0.44320 + (1 - 0.5)^2 0.0]$
$= 9.71$

It is vital to remember that the present value is over two periods, hence the discount factor is $[1/(1 + 0.05)]^2$.

The put option value can be computed using this formula:
$p = PV[\pi^2 p^{++} + 2\pi(1 - \pi)p^{+-} + (1 - \pi)^2 p^{--}]$
The put option value can also be computed simply by applying put–call parity or
$p = c + PV(X) - S = 9.71 + [1/(1 + 0.05)]^2 50 - 50 = 5.06$.
Thus, the current put price is €5.06.

5. Two-Period Binomial Model (Put Options)

Valuation of American Options

American options can be exercised any time before expiration. It is important to remember that American call options on non-dividend paying stocks will not be exercised early. If the underlying stock is expected to decline in value, it is not logical to exercise the call and receive the stock. It is better to sell the call in this situation.

On the other hand, it might make sense to exercise deep in-the-money American put options. This is because the sales proceeds can be invested at the risk-free rate and the interest could be worth more than the time value of the put.

When early exercise has value, the no-arbitrage approach is the only way to value American-style options. As shown in the example below, this approach requires that one works backward through the binomial tree and determines whether early exercise has value at each step.

Example: Valuing a European put and an American put option using the two-period binomial model
(*This is based on Example 6 of the curriculum.*)

Suppose you are given the following information: S_0 = 26, X = 25, u = 1.466, d = 0.656, n = 2 (time steps), r = 2.05% (per period), and no dividends.

1. What is the value of a European put?
2. What is the value of an American put?
3. What is the early exercise premium of the American put?

Solution:
To calculate the put values we should create the binominal tree as shown below:

Item	Value
Underlying	26
Put	~~4.01174~~ 4.28143
Hedge Ratio	~~–0.35345~~ –0.37721

Item	Value
Underlying	38.116
Put	0
Hedge Ratio	0

Item	Value
Underlying	17.056
Put	~~7.44360~~ 7.94400
Hedge Ratio	–0.99970

Item	Value
Underlying	55.87806
Put	0

Item	Value
Underlying	25.00410
Put	0

Item	Value
Underlying	11.18874
Put	13.81126

We need to start at the end of the tree (Time 2) and work backwards. Node 2-1 represents two up moves of the underlying so the value is 26 x 1.466 x 1.466 = 55.88. The value of the put option (whether European or American) is 0. Similarly:

Node 2-2: Underlying = 26 x 1.466 x 0.656 = 25. European put = American put = 0.
Node 2-3: Underlying = 26 x 0.656 x 0.656 = 11.19. European put = American put = 0.
Node 1-1: Underlying = 26 x 1.466 = 38.12. European put = American put = 0.
Node 1-2: Underlying = 26 x 0.656 = 17.06. European put = 7.44, American put = 7.94.

At Node 1-2, values of the European put and the American put are different. This is because the American put can be exercised early but the European put cannot. The value of the American put at Node 1-2 is X – S = 25 – 17.06 = 7.94. Since this value is more than the value of the European put, it is beneficial to exercise the American put early.
The value of the European option is based on the option values at Nodes 2-2 and 2-3, the risk-neutral probability and the risk-free discount rate.

The RN probability, $\pi = (1 + r - d)/(u - d) = (1 + 0.0205 - 0.656)/(1.466 - 0.656) = 0.45$.
$p^- = PV[\pi p^{+-} + (1 - \pi)p^{--}] = (0 + 0.55 \times 13.81) / 1.0205 = 7.44$.

We now have the information to calculate the European put and American put value at Node 0:
1. European put option = (0 + 0.55 x 7.44) / 1.0205 = 4.01.
2. American put option = (0 + 0.55 x 7.94) / 1.0205 = 4.28.

The early exercise premium = 4.28 – 4.01 = 0.27

6. Two-Period Binomial Model (Role of Dividends)

Dividend Payments and Options

The following segment explains dividend payments within the binomial model. This approach is known as the escrow method. Dividends payments lower the value of the stock which has a negative effect on the value of the call option. For dividend paying instruments, it is assumed that dividends are known; therefore, the underlying instrument is split into two components:

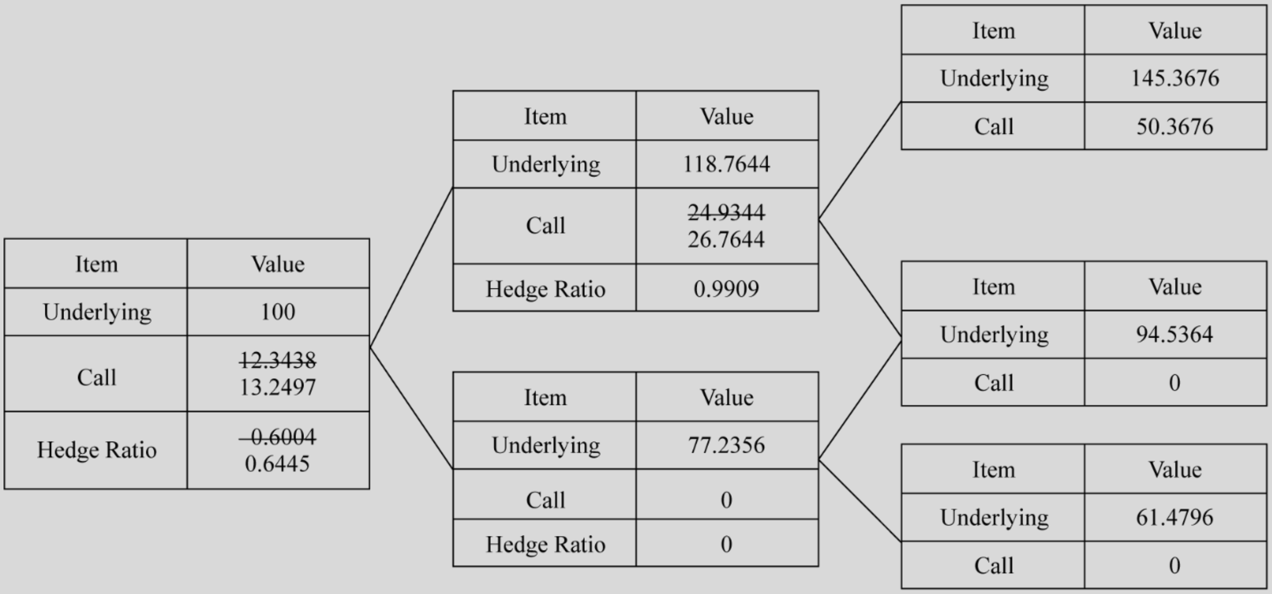
i) the underlying instrument without the known dividends and
ii) the known dividends.

For example, the current value of the underlying instrument without dividends can be expressed as $\hat{S} = S - \gamma$, where γ = the present value of dividend payments and $\hat{S}$ = the underlying instrument without dividends. The uncertainty of the stock is modelled based on $\hat{S}$, not S. At expiration, $\hat{S}_T = S_T$, because we assume any dividends have already been paid. The value of an investment in the stock, however, would be $S_T + \gamma T$, assuming the dividend payments are reinvested at the risk-free rate.

Example: American style call option on stock with dividends

Consider an American call on a US$100 stock with exercise price of US$95. The periodically compounded interest rate is 1.0%, the stock will pay a US$3 dividend at Time 1, u = 1.224, d

= 0.796, and the call option expires in two years.



Here, the first thing to note is that the price of the underlying is modelled ex of dividend. The present value of the dividend at T = 0 is 3/1.01 = 2.97 and the ex-dividend price is 100 - 2.9702 = 97.03.

The up move and down move prices subsequently are calculated using a present price of 97.03. So, S^+ = 97.0297 x 1.224 = 118.7644. S^{++} = 97.03 x 1.224 x 1.224 = 145.37, and so on.

The RN probability of an up move is (1.01 - 0.796)/(1.224 - 0.796) = 0.5

At the up move at T = 1, the binomial model gives a discounted option value of 0.5(50.3676/1.01) + 0.5(0/1.01) = 24.93446. However, since this is an American call, the owner has the option to exercise right before the stock goes ex. Before going ex, the value of the stock would be 118.76 + 3 = 121.76 which would give an option value of 121.76 - 95 = 26.76.

The value of the option at T = 0 is 0.5(26.76/1.01) + 0.5(0/1.01) = 13.25.

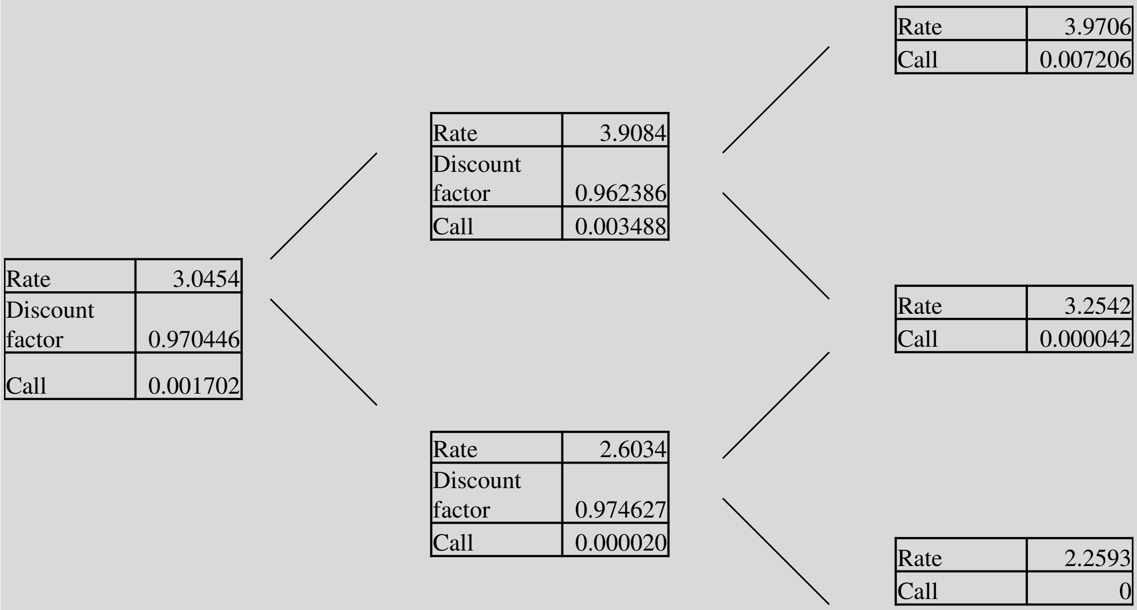
This example illustrates that early exercise option of an American call can have value if the underlying stock pays dividends during the life of contract. Hence, the value of an American call option can be higher than the value of a European call option if the underlying pays dividends.

7. Interest Rate Options and Multiperiod Model

Interest rate options are options where the underlying is an interest rate. A call option on interest rates is in the money when the current spot rate is above the exercise rate. A put option on interest rates is in the money when the current spot rate is below the exercise rate. The example below shows how to value interest rate options using an interest rate tree. The generation of the interest rate tree is outside the scope of this reading. We simply use what is given. With interest rates, we assume that the RN probability of an up move at each node is 50%.

Example: Option on interest rate
(*This is based on Example 8 of the curriculum.*)

Consider a two-year European-style call on the periodically compounded one-year spot interest rate (the underlying). The spot rate is 3.0454%. Assume the notional amount of the options is US$1,000,000 and the call exercise rate is 3.25% of par. Assume the RN probability is 50%. The binomial interest rate tree is given below.



Note that the discount factor at each node depends on the interest rate at that node. Hence, when the up move occurs at T = 1, the discount factor is based on an interest rate of 3.9084%.

At time step 2:
$c^{++} = \text{Max}[0, 3.9706\% - 3.25\%] = 0.007206$. $c^{+-} = \text{Max}[0, 3.2542\% - 3.25\%] = 0.000042$
$c^{--} = \text{Max}[0, 2.2593\% - 3.25\%] = 0$

At time step 1:
$c^{+} = PV_{1,2}[\pi c^{++} + (1 - \pi)c^{+-}]$
$= 0.962386[0.5(0.007206) + (1 - 0.5)0.000042] = 0.003488$
$c^{-} = PV_{1,2}[\pi c^{+-} + (1 - \pi)c^{--}]$
$= 0.974627[0.5(0.000042) + (1 - 0.5)0.0] = 0.00002$

At time step 0:
$c = PV_{rf,0,1}[\pi c^{+} + (1 - \pi)c^{-}]$
$= 0.970446[0.5(0.003488) + (1 - 0.5)0.00002] = 0.00170216$

Since the notional amount is US$1,000,000, the call value is US$1,000,000(0.00170216)] = US$1,702.16

Note: We can use a similar process to calculate the value of put options.

7.1 Multi-period Model

As shown in the previous sections, the two-period model divides the expiration into two periods. Similarly, a three-period model divides expiration into three periods and so on. Thus, with a maturity of T, if there are n time steps, then each time step is T/n in length. As n becomes large, the multi-period mode approaches the BSM model which is discussed in the next section.

For American-style options, we must test, at each node, whether the option is worth more exercised or not exercised.

The no-arbitrage approach can be used to value multi-period European-style or American-style options. The expectations approach can be applied to European-style options but not to American-style options.

8. Black–Scholes–Merton (BSM) Option Valuation Model

8.1 Introductory Material

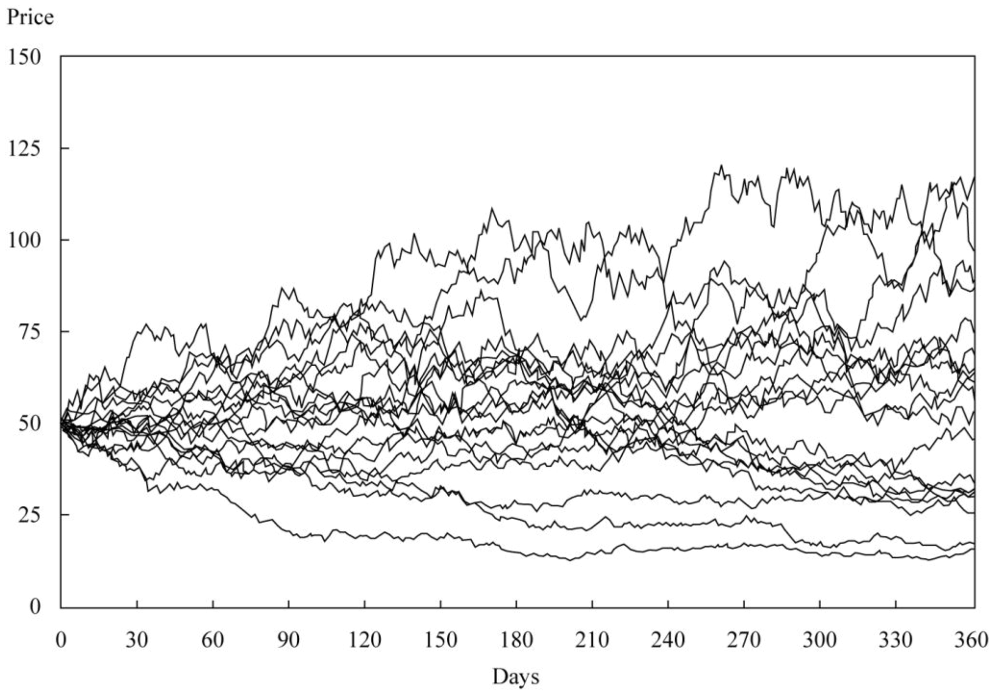
The BSM model is a continuous time version of the discrete time binomial model. It is used to value path-independent options.

8.2 Assumptions of the BSM Model

The key assumptions of the BSM model are:

- The underlying follows a statistical process called geometric Brownian motion (GBM), which implies a lognormal distribution of the return, meaning that the logarithmic return, which is the continuously compounded return, is normally distributed.
- Geometric Brownian motion implies continuous prices, meaning that the price of underlying instrument does not jump from one value to another; rather, it moves smoothly from value to value.
- When price levels are higher there is more variability on an absolute basis as compared to when the price levels are lower. This can be understood by a simple example: a 10% move in a stock with a price of 100 is 10 whereas a 10% move in a stock with a price of 10 is only 1. GBM can never be zero or go below it.

Exhibit 12 illustrates GBM, assuming the initial stock price is S = 50. We assume the stock will grow at 3% (μ = 3% annually, geometrically compounded rate). This GBM process also reflects a random component that is determined by a volatility (σ) of 45%.



Other assumptions of the BSM model are:

- The underlying instrument is liquid, easily traded.
- Continuous trading is available, meaning able to trade at any moment.
- Short selling of the underlying instrument with full use of the proceeds is permitted.
- There are no market frictions, such as transaction costs, regulatory constraints, or taxes.
- No-arbitrage opportunities available.
- All options are European-style, early exercise is not allowed.
- The continuously compounded risk-free interest rate is known and constant; borrowing and lending is at the risk-free rate.
- The volatility of the underlying return is known and constant.
- In case the underlying instrument pays a yield, it is expressed as a continuous known and constant yield at an annualized rate.

9. BSM Model: Components

The BSM model is based on continuous time and therefore uses a continuously compounded interest rate rather than a discretely compounded rate. The interest rate, r, is the annualized continuously compounded rate. The volatility, σ, is the annualized standard deviation in percentage terms.

The BSM model for non-dividend paying stock is:

$c = SN(d_1) - e^{-rT}XN(d_2)$

$p = e^{-rT}XN(-d_2) - SN(-d_1)$

where:

$$d_1 = \frac{\ln(S/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

N(x) denotes the standard normal cumulative distribution function - the probability of obtaining a value of less than x based on a standard normal distribution. Here, x will have value of either d_1 or d_2. N(x) values can be estimated by a computer program or from a statistical table. The standard normal distribution is a normal distribution with a mean of 0 and a standard deviation of 1. S represents the price of the underlying and X is the strike price. T is the time to expiration in years and r is the continuously compounded risk-free rate. Hence, e^{-rT} is a present value factor.

The BSM model can be interpreted in multiple ways as indicated below.

Interpretation 1:
The BSM model can be given as the present value of the expected option payoff at expiration. For calls $c = PV_r[E(c_T)]$ and for puts $p = PV_r[E(p_T)]$
where:
$E(c_T) = Se^{rT}N(d_1) - XN(d_2)$ and $E(p_T) = XN(-d_2) - Se^{rT}N(-d_1)$.
Here, e^{-rT} is the present value term.

Interpretation 2:
The BSM model call value can be interpreted as a stock component, $SN(d_1)$, minus a bond component, $e^{-rT}XN(d_2)$. This is visible in the call option formula: $c = SN(d_1) - e^{-rT}XN(d_2)$.

The BSM model put value can be interpreted as a bond component, $e^{-rT}XN(-d_2)$, minus a stock component, $SN(-d_1)$. This is visible in the put option formula: $p = e^{-rT}XN(-d_2) - SN(-d_1)$.

Taking this concept one step further, an option can be thought of as a dynamically managed portfolio of the underlying stock and zero-coupon bonds. To reproduce the option payoffs with stocks and bonds, the initial cost of this replicating strategy = $n_S S + n_B B$.

Replicating Strategy Cost = $n_S S + n_B B$

where n_s is the number of underlying shares and $n_S = N(d_1) > 0$ for calls and $n_S = -N(-d_1) < 0$ for puts.

The number of bonds is $n_B = -N(d_2) < 0$ for calls and $n_B = N(-d_2) > 0$ for puts.

If n is positive, we are buying and if n is negative, we are selling (short selling).

The price of the zero-coupon bond price, $B = e^{-rT}X$.

For calls, the stock is bought because $n_S = N(d_1) > 0$ and the bond is sold because $n_B = -N(d_2) < 0$. Selling a bond is the same as borrowing money. Therefore, a call option can be viewed as a leveraged position in the stock.

For put options, the bond is bought because $n_B = N(-d_2) > 0$ and shares of the underlying stock are sold because $n_S = -N(-d_1) < 0$. Buying a bond is the same as lending money. A put can be viewed as buying a bond where this purchase is partially financed by short selling the underlying stock.

The following table compares the BSM model with the Binomial model covered earlier.

	Call option		**Put option**	
	Underlying	**Financing**	**Underlying**	**Financing**
Binomial model	hs	$PV(-hS^- + c^-)$	hS	$PV(-hS^- + p^-)$
BSM model	$N(d_1)S$	$-N(d_2)e^{-rT}X$	$-N(-d_1)S$	$N(-d_2)e^{-rT}X$

$N(d_1)$ can be interpreted as the hedge ratio for call options.
$N(d_2)$ can be interpreted as the probability that call option will expire in the money.
$1 - N(d_2) = N(-d_2)$ can be interpreted as the probability that put option will expire in the money.

Instructor's Note:
The following tips will help you remember the formulas.
$c = SN(d_1) - e^{-rT}XN(d_2)$
$p = e^{-rT}XN(-d_2) - SN(-d_1)$

- A call option is of the form S – X, whereas a put option is of the form X – S.
- The present value of strike price X is obtained by multiplying it by e^{-rT}.
- d_1 is associated with S, whereas d_2 is associated with X.
- For call options, we use positive values of d_1 and d_2, whereas for put options we use negative values of d_1 and d_2.

It is unlikely that you will be tested on the formulas for d_1 and d_2.

Example: BSM Model interpretation
(*This is based on example 10 of the curriculum.*)

Suppose we are given the following information on call and put options on a stock: S = 100, X = 100, r = 5%, T = 1.0, and σ = 30%. Thus, based on the BSM model, it can be demonstrated that PV(X) = 95.123, $d_1 = 0.317$, $d_2 = 0.017$, $N(d_1) = 0.624$, $N(d_2) = 0.507$, $N(-d_1) = 0.376$, $N(-d_2) = 0.493$, c = 14.23, and p = 9.35.

1. Using the no-arbitrage approach, how can a call option be replicated?
2. Using the no-arbitrage approach, how can a put option be replicated?

Solution:
The no-arbitrage approach to replicating the **call option** involves purchasing $n_S = N(d_1) = 0.624$ shares of stock partially financed with $n_B = -N(d_2) = -0.507$ shares of zero-coupon bonds priced at $B = Xe^{-rT} = 95.123$ per bond. By definition, the cost of this replicating

strategy is $n_S S + n_B B = 0.624(100) + (-0.507)95.123 = 14.17$, which is the option value.

The no-arbitrage approach to replicating the **put option** involves trading $n_S = -N(-d1) = -0.376$ shares of stock. The negative number means we need to short sell 0.376 shares. We also need to buy $n_B = N(-d_2) = 0.493$ shares of the zero-coupon. Again, the cost of the replicating strategy is $n_S S + n_B B = -0.376(100) + (0.493)95.123 = 9.30$ which is the value of the put option.

The discussion above is about the initial trading strategy. As the price of the underlying changes, the values of n_S and n_B also need to be adjusted. This is called dynamic hedging.

10. BSM Model: Carry Benefits and Applications

Carry benefits include dividends for stock options, foreign interest rates for currency options, and coupon payments for bond options. Under the BSM model, these carry benefits are taken as a continuous yield, denoted as γ^c or simply γ. The carry benefit adjusted BSM model is:

$c = Se^{-\gamma T}N(d_1) - e^{-rT}XN(d_2)$

and

$p = e^{-rT}XN(-d_2) - Se^{-\gamma T}N(-d_1)$

where:

$d_1 = \frac{ln(S/X) + (r - \gamma + \sigma^2/2)T}{\sigma\sqrt{T}}$

$d_2 = d_1 - \sigma\sqrt{T}$.

Instructor's Note:
The formulas for call and put options are similar to the formulas without carry benefit. The stock price is adjusted downwards by multiplying it with $e^{-\gamma T}$. The bond component remains unchanged.
Both d_1 and d_2 go down because of the adjustment made for carry benefits.

The carry benefit adjusted BSM model has two components, a stock component and a bond component.

For call options: $Se^{-\gamma T}N(d_1)$ = stock component, and $e^{-rT}XN(d_2)$ = bond component.
For put options: $Se^{-\gamma T}N(-d_1)$ = stock component, and $e^{-rT}XN(-d_2)$ = bond component.

Both d_1 and d_2 are reduced by carry benefits. An increase in carry benefits lowers the call option value and raises the put option value.

Since carry benefits lower d_1 and d_2 values, the probability of being in the money with call options declines as carry benefit increases.

For stock options, $\gamma = \delta$ which is the continuously compounded dividend yield.

With dividend paying stocks, the arbitrageur is able to receive the benefits of dividend payments when long the stock and has to pay dividends when short the stock. The following table helps understand the effects of dividends on a dynamically managed portfolio of the stock and zero-coupon bonds.

Call option		**Put option**	
Underlying	**Financing**	**Underlying**	**Financing**
$N(d_1)\, e^{-\delta T} S$	$-N(d_2)e^{-rT}X$	$-N(-d_1)\, e^{-\delta T} S$	$N(-d_2)e^{-rT}X$

Dividends influence a dynamically managed portfolio. Higher dividends will lower the value of d_1, thus lowering $N(d_1)$ and $N(d_2)$.

Therefore, for call options higher dividends mean:

- Fewer shares to buy
- Fewer bonds to short sell

Similarly, for put options higher dividends mean:

- More shares to short sell
- More bonds to buy

Example: BSM Model Applied to Equities

(*This is based on example 11 of the curriculum.*)

Suppose we are given the following information on an underlying stock and options: S = 60, X = 60, r = 2%, T = 0.5, δ = 2%, and σ = 45%. Assume we are examining European-style options.

1. Which answer *best* describes how the BSM model is used to value a call option with the parameters given?

A. The BSM model call value is the exercise price times $N(d_1)$ less the present value of the stock price times $N(d_2)$.

B. The BSM model call value is the stock price times $e^{-\delta T}N(d_1)$ less the exercise price times $e^{-rT}N(d_2)$.

C. The BSM model call value is the stock price times $e^{-\delta T}N(-d_1)$ less the present value of the exercise price times $e^{-rT}N(-d_2)$.

2. Which answer *best* describes how the BSM model is used to value a put option with the parameters given?

A. The BSM model put value is the exercise price times $N(d_1)$ less the present value of the stock price times $N(d_2)$.

B. The BSM model put value is the exercise price times $e^{-\delta T}N(-d_2)$ less the stock price times $e^{-rT}N(-d_2)$.

C. The BSM model put value is the exercise price times $e^{-rT}N(-d_2)$ less the stock price times

$e^{-\delta T}N(-d_1)$.

3. Suppose now that the stock does not pay a dividend—that is, $\delta = 0\%$. Identify the correct statement.

A. The BSM model option value is the same as the previous problems because options are not dividend adjusted.
B. The BSM model option values will be different because there is an adjustment term applied to the exercise price, that is $e^{-\delta T}$, which will influence the option values.
C. The BSM model option value will be different because d_1, d_2, and the stock component are all adjusted for dividends.

Solution to 1:
B is correct. The BSM call model for a dividend-paying stock can be expressed as $Se^{-\delta T}N(d_1) - Xe^{-rT}N(d_2)$.

Solution to 2:
C is correct. The BSM put model for a dividend-paying stock can be expressed as $Xe^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1)$.

Solution to 3:
C is correct. The BSM model option value will be different because d_1, d_2, and the stock component are all adjusted for dividends.

BSM Model and Currency Options
For foreign exchange options, we will use the price currency / base currency format. If the PKR-USD exchange rate is 100 PKR to 1 USD, this will be written as 100 PKR/USD. Here USD is the base currency and PKR is the price currency. The carry benefit is the continuously compounded risk-free interest rate for the base currency, $\gamma = r^f$.

With currency options, the underlying and the exercise price must be quoted in the same currency unit. The volatility in the model is the volatility of the log return of the spot exchange rate.

The value of a currency call option is given by:
$c = Se^{-r^f T}N(d_1) - e^{-rT}XN(d_2)$

Here, S is the currency exchange rate in the price currency/base currency format, r^f is risk-free rate in the base currency, and r is the risk-free rate in the price currency.

The BSM call model for currencies can be interpreted as the:
foreign exchange component, $Se^{-r^f T}N(d_1)$, minus the bond component, $e^{-rT}XN(d_2)$.

The value of a currency put option is given by:
$p = e^{-rT}XN(-d_2) - Se^{-r^f T}N(-d_1)$

The BSM put model for currencies can be interpreted as the:
bond component, $e^{-rT}XN(-d_2)$, minus the foreign exchange component, $Se^{-r_f T}N(-d_1)$.

Instructor's Note:
The formulas are similar to the formulas for stocks with dividends. Here, instead of dividends, the benefit is the interest rate on the foreign currency. The spot rate S is adjusted by multiplying it with $e^{-r_f T}$.

Example: BSM model on currency options
(*This is based on example 13 of the curriculum.*)

A Japanese camera exporter to Europe has contracted to receive fixed euro (€) amounts each quarter for his goods. The spot price of the currency pair is 135¥/€. The exporter is concerned that the yen will strengthen because in this case, his forthcoming fixed euro will buy fewer yen. Hence, the exporter is considering buying an at-the-money spot euro put option to protect against this fall; this in essence is a call on yen. The Japanese risk-free rate is 0.25% and the European risk-free rate is 1.00%.

1. What are the underlying and exercise prices to use in the BSM model to get the euro put option value?
2. What are the risk-free rate and the carry rate to use in the BSM model to get the euro put option value?

Solution to 1: The underlying is the spot FX price of 135 ¥/€. Because the put is at-the-money spot, the exercise price equals the spot price.

Solution to 2: Recognize that Yen is the price currency and euro is the base currency. The risk-free rate to use is the Japanese rate, i.e., 0.25% because Yen is the price economy. The carry rate is the base currency's risk-free rate, which is the European rate, i.e., 1.00%.

11. Black Option Valuation Model and European Options on Futures

Fischer Black introduced a modified version of the BSM model that is applicable to options on underlying instruments such as options on futures contracts, interest rate-based options, and swaptions.

11.1 European Options on Futures

The underlying instrument is the futures contract – for example equity index futures – and the assumption is that the futures price follows geometric Brownian motion. Margin requirements and marking to market are ignored.

$F_0(T)$ = the futures price at Time 0 that expires at Time T,
σ = volatility related to the futures price.
X = exercise price.

e^{-rT} = present value term.
Other terms are as defined previously.

We need to be aware of two interpretations of the Black model:
Interpretation 1:

- c = present value of difference between futures price and exercise price adjusted by N(d) functions.
- p = present value of difference between exercise price and futures price adjusted by N(d) functions.

Interpretation 2:
The option value obtained through the Black model has two components, a futures component and a bond component.

- c = futures component, $F_0(T)e^{-rT}N(d_1)$, minus the bond component, $e^{-rT}XN(d_2)$
- p = bond component, $e^{-rT}XN(-d_2)$, minus the futures component, $F_0(T)e^{-rT}N(-d_1)$

Futures Put-Call Parity under the Black Model
According to the put-call parity, protective put = fiduciary call
i.e., S + P = C + PV(X)

Futures option put–call parity under the Black model is given as:
$PV[F_0(T)] + p = c + PV(X)$
i.e., $e^{-rT} F_0(T) + p = c + e^{-rT} X$
$c = e^{-rT}[F_0(T) - X] + p$

Example: The Black Model on Options on futures
(*This is based on Example 14 of the curriculum.*)

The S&P 500 Index (a spot index) is presently at 1,860 and the 0.25 expiration futures contract is trading at 1,851.65. Suppose further that the exercise price is 1,860, the continuously compounded risk-free rate is 0.2%, time to expiration is 0.25, volatility is 15%, and the dividend yield is 2.0%. Based on this information, the following results are obtained for options on the futures contract.

Calls	**Puts**
$N(d_1) = 0.491$	$N(-d_1) = 0.509$
$N(d_2) = 0.461$	$N(-d_2) = 0.539$
c = US$51.41	p = US$59.76

1. To value a European call using the Black model, what values would the present value of exercise price and present value of futures price be multiplied with?
2. To value a European put using the Black model, what values would the present value of exercise price and present value of futures price be multiplied with?
3. What are the underlying and exercise prices to use in the Black futures option model?

Solution to 1: The present value of the futures value would be multiplied with $N(d_1)$, 0.491 and the present value of the exercise price would be multiplied with $N(d_2)$, 0.461.

Solution to 2: The present value of the futures value would be multiplied with $N(-d_1)$, 0.509 and the present value of the exercise price would be multiplied with $N(-d_2)$, 0.539.

Solution to 3: The underlying is the futures price of 1,851.65 and the exercise price was given as 1,860.

12. Interest Rate Options

An interest rate call option gives the call holder the right to a certain cash payment when the underlying interest rate is more than the exercise rate. An interest rate put option gives the put holder the right to a certain cash payment when the underlying interest rate is less than the exercise rate.

Consider an interest rate call option with one-year expiration. The underlying interest rate is a forward rate agreement (FRA) that expires in one year and is based on three-month Libor. This FRA observed today is the underlying rate in the Black model.

Graphically:

c = call option expires in one-year
3-month FRA expires in one year

t_{j-1} = time to option expiration = when interest rate is set = 1 year in the above example
t_m = Libor deposit maturity = 3 months in this example = 0.25 years

Under the standard market model, the interest rate call value is given by:

$c = (AP)e^{-r(t_{j-1} + t_m)}[FRA(0, t_{j-1}, t_m) N(d_1) - R_X N(d_2)]$

plugging in the time we get:

$c = (AP)e^{-r(1.25)}[FRA(0,1,0.25)N(d_1) - R_X N(d_2)]$

AP = Accrual period; for 90-day Libor the AP is 90/360
FRA(0,1,0.25) = FRA rate at Time 0 that expires at Time 1 and is based 0.25 year Libor
R_X = Exercise rate

Under the standard market model, the interest rate put value is given by:
$p = (AP)\, e^{-r(t_{j-1} + t_m)}[R_X N(-d_2) - FRA(0, t_{j-1}, t_m))N(-d_1)]$

For both the put and call formulas:

$d_1 = \dfrac{ln\left[\frac{FRA(0,tj-1,tm)}{R_X}\right]+\left(\frac{\sigma^2}{2}\right)t_{j-1}}{\sigma\sqrt{t_{j-1}}}$ and $d_2 = d_1 - \sigma\sqrt{t_{j-1}}$

All interest rates are stated on an annual basis. The volatility, σ, refers to interest rate volatility. The put and call values are based on $1 notional amount.

As before, we need to be aware of two interpretations:

Interpretation 1:
The standard market model can also be given as the present value of the expected option payoff at expiration.

- $c = PV[E(c)]$ where $E(c) = (AP)[FRA(0, t_{j-1}, t_m)N(d_1) - R_X N(d_2)]$
- $p = PV[E(p)]$ where $E(p) = (AP)[R_X N(-d_2) - FRA(0, t_{j-1}, t_m)N(-d_1)]$

Interpretation 2:
The option value has two components, a futures component and a bond component.

- Call value = FRA component minus the bond component
- Put value = bond component minus the FRA component

The standard market model has the same general form as the Black model, but there are important differences. The differences with the standard market model are:

1. The discount factor does not apply to the option expiration, t_{j-1}, as in in the case of the Black model, but applies to the maturity of the underlying FRA or t_j (= $t_{j-1} + t_m$).
2. The underlying is an interest rate (FRA), not a futures price.
3. The exercise price is an interest rate (R_x), not a price.
4. The time to option expiration, t_{j-1}, is used in the calculation of d_1 and d_2.
5. Finally, both the forward rate and the exercise rate should be expressed in decimal form and not as a percentage. Or if expressed as a percentage, then the notional amount adjustment should be divided by 100.

Example: Interest rate options
(*This is based on Example 15 of the curriculum.*)

On 15 May, an investor anticipates that he would need to borrow 10,000,000 Singapore dollars on 15 June to fund the purchase of an asset, which he expects to sell after three months on 15 September. The current three-month Sibor (that is, Singapore Libor) is 0.55%. The appropriate FRA rate over the period of 15 June to 15 September is currently 0.68%. The interest rate call option has an exercise rate of 0.60%.

1. In using the Black model to value this interest rate call option, what would the underlying rate be?
2. The discount factor used in pricing this option would be over what period of time?

Solution to 1: In using the Black model, a forward or futures price is used as the underlying. Therefore, the current FRA rate of 0.68% is the underlying.

Solution to 2: The discount factor would be from 15 May – 15 September. This is because the value of the option is determined on 15 May whereas the payoff will occur on 15

September.

Combinations using Interest Rate Options

The following combinations can be created using interest rate options.

1. If exercise rate = current FRA rate, then long interest rate call option + short interest rate put option = receive-floating, pay-fixed FRA.
2. If exercise rate = current FRA rate, then long interest rate put option + short interest rate call option = receive-fixed, pay-floating FRA.
3. A set of floating-rate loan payments can be hedged with a long position in an interest rate cap encompassing a series of interest rate call options.
4. A floating-rate lending situation can be hedged with an interest rate floor encompassing a series of interest rate put options.
5. Long interest rate cap + short interest rate floor with the same exercise rate = receive-floating, pay-fixed interest rate swap
6. Long interest rate floor + short interest rate cap with the same exercise rate = a receive-fixed, pay-floating interest rate swap
7. If the exercise rate is set equal to the swap rate, then the value of the cap must be equal to the value of the floor.

13. Swaptions

A swaption is an option on a swap. It gives the holder the right, but not the obligation, to enter a swap at a pre-agreed swap rate, the exercise rate. There are two types of swaptions.

1. **A payer swaption** is an option on a swap to pay fixed, receive floating.
2. **A receiver swaption** is an option on a swap to receive fixed, pay floating.

The payer swaption valuation model for \$1 notional amount is:

$PAY_{SWN} = (AP)PVA[R_{FIX}N(d_1) - R_XN(d_2)]$

Here AP stands for the accrual period. If a swap is settled every quarter, AP = 90/360. PVA refers to the present value of an annuity. R_{FIX} is the market swap fixed rate at expiration of the option. R_X is the exercise rate.

A payer swaption has a positive value if the market swap fixed rate at expiration is higher than the exercise rate. Say a company enters into a payer swaption which expires in two years. The underlying is a three-year annual pay swap with a fixed rate of 5% (R_X). When the swaption expires in two years, the fixed rate of a three-year annual pay swap is 6% (R_{FIX}). The company has benefited in this case because it can now enter into a three-year swap at a 5% fixed rate even though the market rate is 6%. This benefit can be expressed as an annuity. The PVA function allows us to calculate the present value of that annuity.

The receiver swaption valuation model for \$1 notional amount is:

$REC_{SWN} = (AP)PVA[R_XN(-d_2) - R_{FIX}N(-d_1)]$

A receiver swaption has a positive value if the market swap fixed rate at expiration is less than then the exercise rate.

The formulas for d_1 and d_2 are:

$d_1 = \frac{\ln\left(\frac{R_{FIX}}{R_X}\right)+\left(\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$

Here, σ is the volatility of the forward swap rate.

Two interpretations of swaptions model

Interpretation 1:
The swaption model is the present value of the expected option payoff at expiration.

- $PAY_{SWN} = PV[E(PAY_{SWN})]$ where $E(PAY_{SWN}) = e^{rT}PAY_{SWN}$
- $REC_{SWN} = PV[E(REC_{SWN})]$ where $E(REC_{SWN}) = e^{rT}REC_{SWN}$

Interpretation 2:
The swaption value has two components, a swap component and a bond component.

- Payer swaption value = the swap component - the bond component
  In symbols: $PAY_{SWN} = (AP)PVA(R_{FIX})N(d_1) - (AP)PVA(R_X)N(d_2)$
- Receiver swaption value = the bond component - the swap component
  In symbols: $REC_{SWN} = (AP)PVA(R_X)N(-d_2) - (AP)PVA(R_{FIX})N(-d_1)$

The swaption model looks like the Black model but there are important differences:

1. The discount factor is absent. The payoff is not a single payment but a series of payments. PVA (present value of an annuity) embeds the discount factor.
2. The underlying is the fixed rate on a forward interest rate swap.
3. The exercise price is an interest rate.
4. Both the forward swap rate and the exercise rate should be expressed in decimal form and not as percent.

Example: Swaptions
(*This is based on Example 16 of the curriculum.*)

Suppose you are an Australian company and have ongoing floating-rate debt. You have profited for some time by paying at a floating rate because rates have been falling steadily for the last few years. Now, however, you are concerned that within three months the Australian central bank may tighten its monetary policy and your debt costs will thus increase. Rather than lock in your borrowing via a swap, you prefer to hedge by buying a swaption expiring in three months, whereby you will have the choice, but not the obligation, to enter a five-year swap locking in your borrowing costs. The current three-month forward, five-year swap rate is 2.65%. The current five-year swap rate is 2.55%. The current three-month risk-free rate is 2.25%.

1. What is the time to expiration for valuing the swaption using the Black Model?

2. What is the risk-free rate for valuing the swaption using the Black Model?

Solution to 1: The time to expiration of the swaption is three months.

Solution to 2: The risk-free rate to use is 2.25%, the three-month risk-free rate.

Equivalence Relationships

The following equivalence relationships hold for swaptions:

1. Long interest rate cap + short interest rate floor with the same exercise rate = receive-floating, pay-fixed interest rate swap
2. Short interest rate cap + long interest rate floor with the same exercise rate = pay-floating, receive-fixed interest rate swap
3. Long receiver swaption + short payer swaption with the same exercise rate = receive-fixed, pay-floating forward swap
4. Long payer swaption + short receiver swaption with the same exercise rate = receive-floating, pay-fixed forward swap
5. Long callable fixed-rate bond = long straight fixed-rate bond + short receiver swaption
6. If the exercise rate is selected such that the receiver and payer swaptions have the same value, then the exercise rate is equal to the at-market forward swap rate (put-call parity relationship).

14. Option Greeks and Implied Volatility: Delta

The inputs to the BSM model are:

- Price of underlying stock, S
- Time to maturity in years, T
- Strike price, X
- Volatility of underlying in annualized percentage terms, σ
- Continuously compounded risk-free rate, r

Option Greeks measure how much the option value will change for a small change in a particular input. The measures examined here are delta, gamma, theta, vega, and rho. European stock options in which the underlying stock is assumed to pay a dividend yield (denoted by δ) are considered here. For non-dividend-paying stocks, δ = 0.

14.1 Delta

Delta is defined as the change in price of an instrument for a given small change in the value of the stock, holding all else constant.

By definition, the delta of a long position in a stock is 1. The delta of short position in a stock is –1.

An option delta is the change in an option value for a given small change in the value of the underlying stock, holding all else constant.

The option deltas for calls: $\text{Delta}_c = e^{-\delta T}N(d_1)$

The option deltas for puts: $\text{Delta}_p = -\, e^{-\delta T}N(-d_1)$

In the above expressions, δ is the dividend yield on the underlying stocks. For non-dividend paying stocks δ is 0 and $e^{-\delta T}$ is 1.

Deltas change with changes in stock price and time to maturity. When the stock price increases, the call option goes deeper in the money and the value of $N(d_1)$ moves towards 1. As the stock price decreases, the call option goes deeper out of the money and the value of $N(d_1)$ moves toward zero. Closer to option maturity, delta drifts either towards 0 (if out of the money) or towards 1 (if in the money).

Delta hedging an option refers to establishing a position in an option and the underlying stock such that there is no exposure to small changes in the stock price. A related term is 'delta neutral portfolio'. This refers to a portfolio with a delta of 0. To create a delta neutral portfolio, the first step is to identify the hedging instrument. The table identifies possible hedging instruments:

Original Portfolio	**Possible Hedging Instrument**
Long stock	Short call
Long call	Short stock
Long put	Long stock

Once a hedging instrument has been identified, the next step is to calculate the number of hedging instruments required. If Delta_H is the delta of the hedging instrument, the optimal number of units of the hedging instruments, N_H, is given by the formula below:

$$N_H = \frac{-\text{ Original Portfolio delta}}{\text{Delta}_H}$$

If N_H = negative, then short the hedging instrument.
If N_H = positive, then go long/buy the hedging instrument.

For example:

i. If a portfolio consists of 100,000 shares of ABC stock, the portfolio delta is 100,000. How can this portfolio be hedged using stocks?
   $\text{Delta}_H = 1$.
   $N_H = -100,000/1 = -100,00$. Hence, short 100,000 shares.

ii. How can this portfolio be hedged using call options with a delta of 0.5?
   $\text{Delta}_H = 0.5$.
   $N_H = -5,000/0.5 = -10,000$. Hence, short 10,000 call options.

iii. A portfolio of call options has a delta of -1,500.
   We want to hedge using the underlying stock. $\text{Delta}_H = 1$.

$N_H = -(-1{,}500)/1 = 1{,}500$. Hence, buy 1,500 shares of stock.

Example: Delta Hedging
(*This is based on Example 17 of the curriculum.*)

Apple stock is trading at US$125. We write calls (that is, we sell calls) on 1,000 Apple shares and now are exposed to an increase in the price of the Apple stock. We want to neutralize our exposure to Apple. Say the call delta is 0.50. Hence, we buy 500 Apple shares to hedge. Now, if Apple goes up US$0.10, the sold calls will go up US$50 (our liability goes up), but our long 500 Apple hedge will profit by US$50. Hence, we are delta hedged.
Identify the *incorrect* statement:

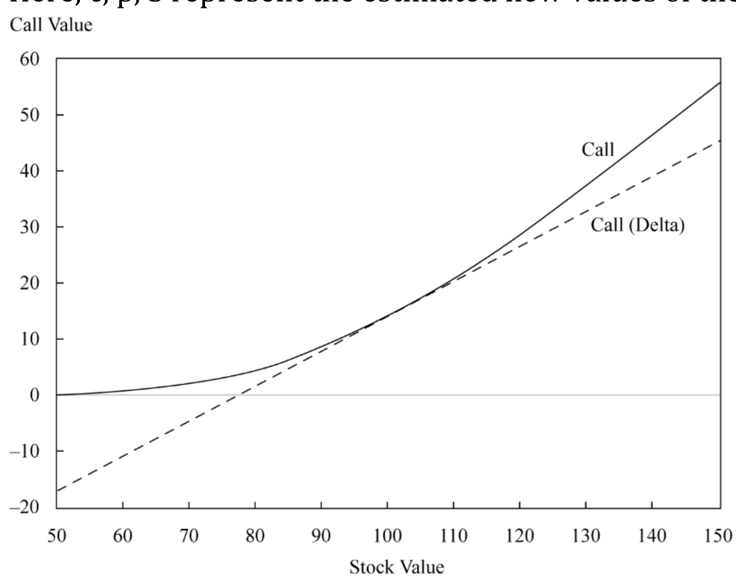
A. If we sell Apple puts, we need to buy Apple stock to delta hedge.
B. Call delta is non- negative (≥ 0); put delta is non- positive (≤ 0).
C. Delta hedging is the process of neutralizing exposure to the underlying.

Solution:
A is the correct answer because statement A is incorrect. If we sell puts, we need to short sell stock to delta hedge.

Another Interpretation of Delta
The delta measure can be used to estimate the change in value of an option when the price of the underlying changes. The relevant formulas are given below:

$\hat{c} - c \cong \text{Delta}_c (\hat{S} - S)$
$\hat{p} - p \cong \text{Delta}_p (\hat{S} - S)$

Here, $\hat{c}$, $\hat{p}$, $\hat{S}$ represent the estimated new values of the call, put, and stock.



A simple numerical example will illustrate these formulas. Say a stock is trading at $100. A call option with a delta of 0.5 is currently priced at $5. A put option with a delta of -0.5 is priced at $4. If the stock price increases to $102, what is the estimated new value of the call option and put option?

Using the call option formula: $\hat{c} - 5 \cong 0.5(102 - 100)$. $\hat{c} = 5 + 1 = 6$.

Using the put option formula: $\hat{p} - 4 \cong -0.5(102 - 100)$. $\hat{p} = 4 - 1 = 3$.

Example: Delta Hedging
(*This is based on Example 18 of the curriculum.*)

Suppose we know S = 100, X = 100, r = 5%, T = 1.0, σ = 30%, and δ = 5%. We have a short position in put options on 10,000 shares of stock. Based on this information, we note Delta_c = 0.532, and Delta_p = –0.419. Assume each stock option contract is for one share of stock.

1. Assuming the hedging instrument is a stock, how many stocks should be bought/sold for a delta hedge?
2. Assuming the hedging instrument is a call option, how many call options should be bought/sold for a delta hedge?
3. Identify the correct interpretation of an option delta.

A. Option delta measures the curvature in the option price with respect to the stock price.
B. Option delta is the change in an option value for a given small change in the stock's value, holding everything else constant.
C. Option delta is the probability of the option expiring in the money.

Solution to 1:
B is correct. $N_H = \frac{-\text{ Original Portfolio delta}}{\text{Delta}_H}$
The put delta is given as –0.419, thus the short put delta is 0.419. In this case, Portfolio delta = 10,000(0.419) = 4,190 and Delta_H = 1.0. Thus, the number of hedging units is –4,190 [= –(4,190/1)] or short sell 4,190 shares of stock.

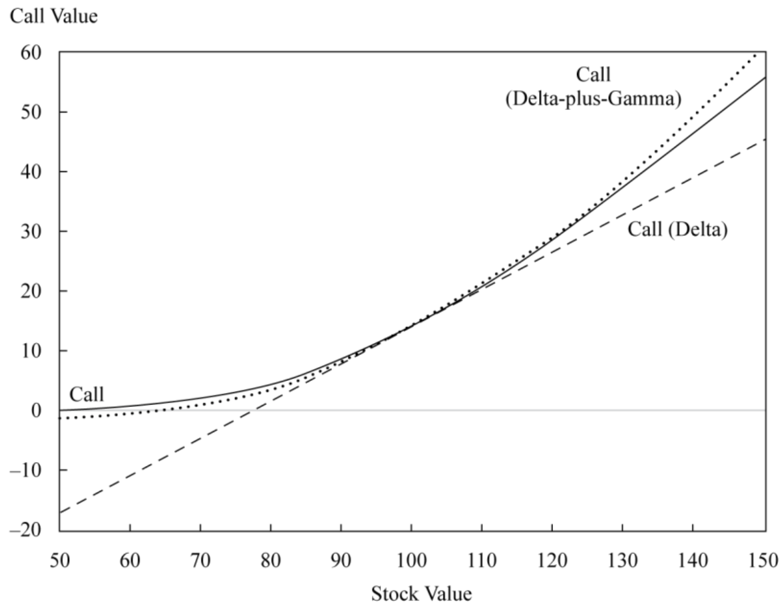
Solution to 2:
A is correct. Again, the Portfolio delta = 4,190 but now Delta_H = 0.532. Thus, the number of hedging units is –7,875.9 [= –(4,190/0.532)] or sell 7,876 call options.

Solution to 3:
B is correct. Delta is defined as the change in a given portfolio for a given small change in the stock's value, holding everything else constant. Option delta is defined as the change in an option value for a given small change in the stock's value, holding everything else constant.

15. Gamma

An option gamma is the change in a given option delta for a given small change in the stock's value, all else constant. Option gamma is a measure of curvature in option price versus the stock value. Based on put-call parity, the gamma for a call and put option are the same:

$\text{Gamma}_c = \text{Gamma}_p = \frac{e^{-\delta T}}{S\sigma\sqrt{T}} n(d_1)$



A few key points related to gamma are as follows:

- Gamma is always non-negative.
- Gamma has its largest value when an option is at the money.
- Gamma changes as the stock price changes and time to expiration changes.
- Gamma approximates the estimation error in delta for options because the option price with respect to the stock is non-linear.
- Gamma measures the non-linearity risk or the risk of the delta neutral portfolio.
- A gamma neutral portfolio means gamma is zero.
- When hedging an option portfolio, the option gamma has to be managed and then the portfolio delta is neutralized.

Another Interpretation of Gamma

In the previous section, we looked at formulas that the estimated the change in option value based on delta. These formulas can be improved by also considering the option gamma. The revised formulas are shown below:

$$\hat{c} - c \approx Delta_c(\hat{S} - S) + \frac{Gamma_c}{2}(\hat{S} - S)^2$$

$$\hat{p} - p \approx Delta_p(\hat{S} - S) + \frac{Gamma_p}{2}(\hat{S} - S)^2$$

Instructor's Note:

These formulas are similar to the duration and convexity adjustment formulas in fixed income.

The discussion above was based on options. The gamma of stock, whether long or short, is always 0. This is because the delta of a stock is always 1 for a long position and -1 for a short position. Since the delta does not change, the gamma is 0.

In reality, stock prices tend to 'jump' suddenly rather than move continuously and smoothly. This creates a 'Gamma Risk', i.e., a risk that stock prices would jump and suddenly leave the position un-hedged.

Example: Gamma Risk in Option Trading
(*This is based on Example 19 of the curriculum.*)

Suppose we are options traders and have only one option position—a short call option. We also hold some stock such that we are delta hedged. Which one of the following statements is true?

A. We are gamma neutral.
B. Buying a call will increase our overall gamma.
C. Our overall position is a positive gamma, which will make large moves profitable for us, whether up or down.

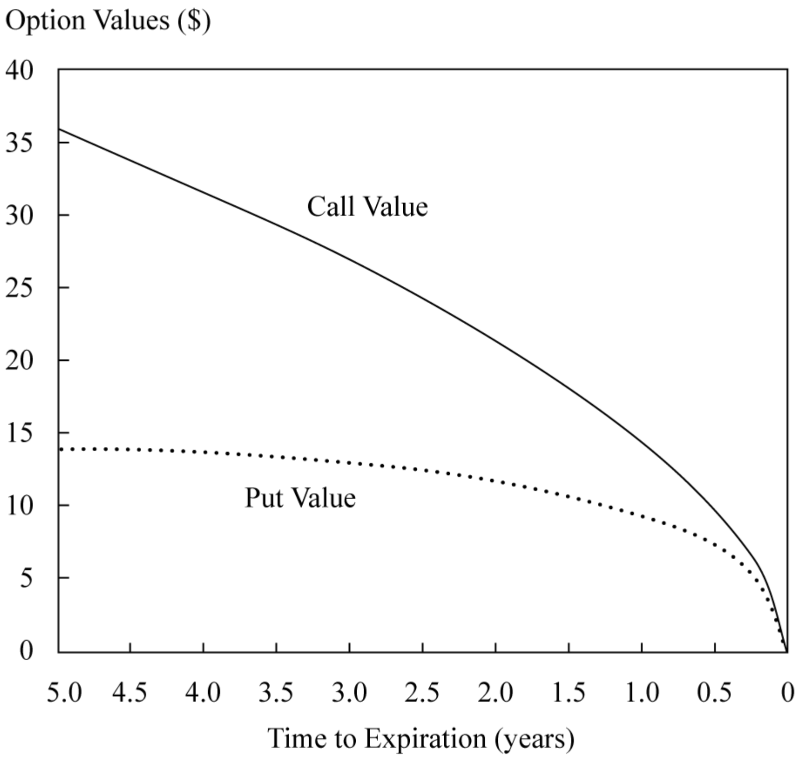
Solution:
B is correct. Buying options (calls or puts) will always increase net gamma. A is incorrect because we are short gamma, not gamma neutral. C is also incorrect because we are short gamma. We can only become gamma neutral from a short gamma position by purchasing options.

16. Theta

Option theta is defined as the change in an option value for a given small change in calendar time, all else constant. It is the rate at which the option time value declines as an option approaches expiration. Because stocks have no expiration, the stock theta is zero.

The gain or loss of an option portfolio/position in response to the mere passage of calendar time is known as time decay. For long options positions, as time elapses, without any change in other variables, there can be significant losses in value. Therefore, investment managers with considerable option positions monitor theta and time decay. Theta is negative for options because as time passes, option expiration declines and so does option value.

Exhibit 17 of the curriculum illustrates the option value with respect to time to expiration for at the money call and put options.



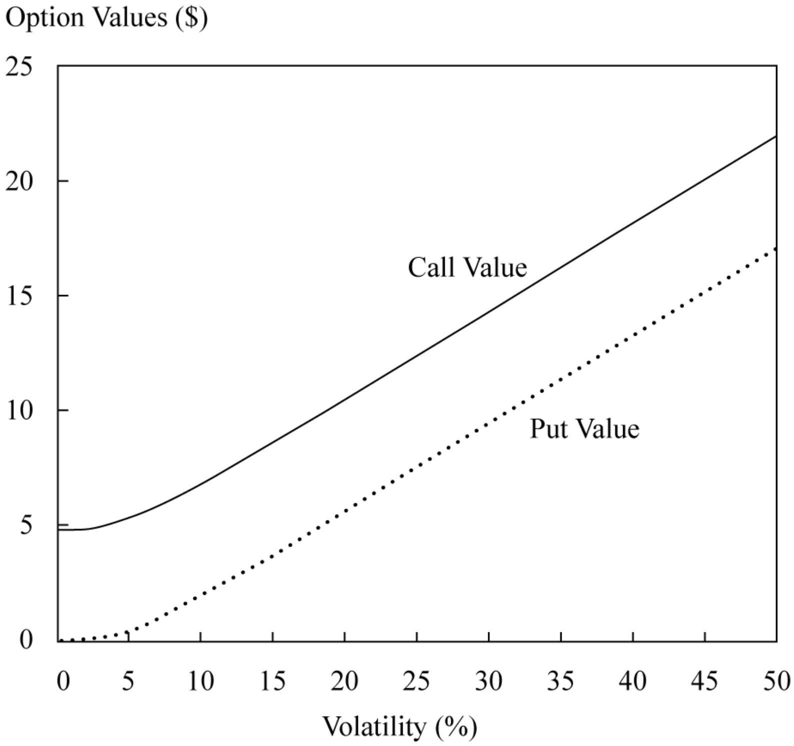
17. Vega

"Vega is defined as the change in a given portfolio for a given small change in volatility, holding everything else constant." Vega measures the sensitivity of portfolio to volatility of the underlying instrument.

A few key points related to vega are as follows:

- The vega of an option is positive because as volatility rises, option value for both calls and puts rises.
- The vega of a call equals the vega of a put based on put–call parity.
- Vega is based on an unobservable parameter, future volatility, which is subjective.
- Vega is high for options that are at or near the money.
- Vega is high for options that are short dated.
- Volatility is hedged with other options.

Exhibit 18 illustrates the option values with respect to volatility.



18. Rho

Rho is the change in a given portfolio for a given small change in the risk-free interest rate, all else constant. It measures the sensitivity of the portfolio with respect to the risk-free interest rate.

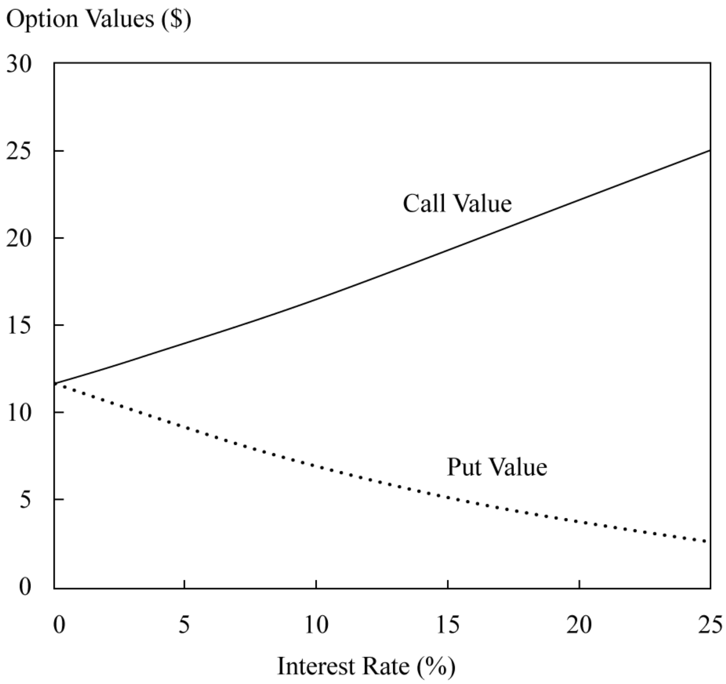
The rho of a call is positive. The buying of an option avoids the costs associated with purchasing stock; hence, a call option allows an investor to earn interest on the money that is not used for buying the stock. The higher the interest rate, the higher the call value.

The rho of a put is negative. Because the option to sell the stock delays the opportunity to earn interest on the sale proceeds, therefore, the higher the interest rate, the lower the put value.

When interest rates are zero, the call and put option values are the same for at-the-money options. As interest rates rise, the difference increases between call and put options.

Rho is low, the impact of changes in interest rate changes to option values is relatively small. Therefore, the influence of interest rates is generally not a major concern for option traders.

Exhibit 19 of the curriculum illustrates the relationship between option values and interest rates.



19. Implied Volatility

Volatility is not observable. It can be estimated based on historical data. For example, for a three-month option, we can calculate the actual historical stock volatility over the last three months and based on this value, give an estimate of volatility over the next three months. The volatility parameter in the BSM model, however, is the *future* volatility. Therefore, the option can be "mispriced" with respect to the actual future volatility.

Different investors have different views of the future volatility. The most accurate estimation of the option value depends upon the most accurate forecast.

Volatility can be inferred from option prices, which is known as the implied volatility. Thus, the BSM model can be inverted to estimate implied volatility. Implied volatility is important as it gives information about the perceived uncertainty in the market and collective opinion of investors on the volatility of the underlying and the demand for options. This kind of information is beneficial to traders.

As implied volatility increases, market participants are communicating an increased market price risk. For example, if the implied volatility of a put increases, the cost of buying a put increases, making hedging expensive.

Implied volatility can be understood as market's view of how to value options. Market participants often quote options in volatility terms. For example, a call option trading for €15.25, may be quoted as 20.00%, where 20.00% is the implied volatility based on a €15.25 option price.

Because volatility has the same magnitude across exercise prices and expiration dates, it can be used to compare the values of two options, with different exercise prices and expiration dates.

Example: Implied Volatility in Option Trading within One Market
(*This is based on Example 20 of the curriculum.*)

Suppose we hold a portfolio of options all tied to FTSE 100 futures contracts. Let the current futures price be 6,850. A client calls to request our offer prices on out-of-the-money puts and at-the-money puts, both with the same agreed expiration date. We calculate the prices to be respectively, 190 and 280 futures points. The client wants these prices quoted in implied volatility as well as in futures points because she wants to compare prices by comparing the quoted implied volatilities. The implied volatilities are 16% for the out-of-the-money puts and 15.2% for the at-the-money puts. Why does the client want the quotes in implied volatility?

Solution:
Implied volatility can be used to assess the relative value of different options, neutralizing the moneyness and time to expiration effects. Also, implied volatility is useful for revaluing existing positions over time.

Example: Implied Volatility
(*This is based on Example 21 of the curriculum.*)

Suppose an options dealer offers to sell a three-month at-the-money call on the FTSE index option at 19% implied volatility and a one-month in-the-money put on Vodaphone (VOD) at 24%. An option trader believes that based on the current outlook, FTSE volatility should be closer to 25% and VOD volatility should be closer to 20%. What actions might the trader take to benefit from her views?

Solution:
The trader should buy the FTSE Index call and sell the Vodaphone put. The trader expects the FTSE volatility to rise and VOD volatility to fall, therefore the FTSE call option's value is expected to rise and the VOD put option's value is expected to fall.

The BSM model assumes that volatility is known and constant and that all investors agree on the value of volatility. In reality, there can be different implied volatilities for calls and puts with the same terms. Implied volatility can vary across exercise prices and the relationship is known as the volatility smile or the skew depending on the particular shape. The implied volatility with respect to time to expiration is called the term structure of volatility. A three-dimensional plot of the implied volatility with respect to both expiration and exercise prices may be constructed which is known as the volatility surface.

Summary

LO: Describe and interpret the binomial option valuation model and its component terms.

The binomial option valuation model is based on the no-arbitrage approach. The arbitrageur seeks mispricing between the option price and the underlying spot price to make profits. In doing so, the arbitrageur follows two rules:

- Rule #1: Do not use your own money – the arbitrageur borrows money to purchase the underlying and invests proceeds from short selling transactions at the risk-free rate.
- Rule #2: Do not take any price risk – the arbitrageur eliminates any market price risk related to the underlying and the derivatives used.

LO: Calculate the no-arbitrage values of European and American options using a two-period binomial model.

The no-arbitrage approach is used for option valuation and is built on the key concept of the law of one price, which says that if two investments have the same future cash flows regardless of what happens in the future, then these two investments should have the same current price. The two-period binomial model can be viewed as three one-period binomial models, one positioned at Time 0 and two positioned at Time 1.

LO: Identify an arbitrage opportunity involving options and describe the related arbitrage.

An arbitrage opportunity exists when there is a difference between option value and its underlying price. The option can be bought (sold) when its value is higher (lower) than its spot price.

LO: Calculate and interpret the value of an interest rate option using a two- period binomial model.

Interest rate option valuation requires the specification of an entire term structure of interest rates. The possible interest rate scenarios are presented using a binomial tree and the value of the option is then worked backwards, starting from the ending node to the beginning node.

LO: Describe how the value of a European option can be analyzed as the present value of the option's expected payoff at expiration.

In general, European-style options can be valued based on the expectations approach in which the option value is determined as the present value of the expected future option payouts, where the discount rate is the risk-free rate and the expectation is taken based on the risk-neutral probability measure.

LO: Identify assumptions of the Black–Scholes–Merton option valuation model.

The assumptions of the BSM model are:

- The underlying follows geometric Brownian motion, implying a lognormal distribution of the return, meaning that the continuously compounded logarithmic return is normally distributed.
- Geometric Brownian motion implies continuous prices, meaning no jumps in the price of underlying instrument only smooth movements from value to value.
- The underlying instrument is liquid and easily traded.
- Continuous trading is available, meaning investors are able to trade at any moment.
- Short selling of the underlying instrument with full use of the proceeds is permitted.
- There are no market frictions, such as transaction costs, regulatory constraints, or taxes.
- No-arbitrage opportunities available.
- All options are European-style, early exercise not allowed.
- The continuously compounded risk-free interest rate is known and constant; borrowing and lending is at the risk-free rate.
- The volatility of the underlying return is known and constant.
- In case the underlying instrument pays a yield, it is expressed as a continuous known and constant yield at an annualized rate.

LO: Interpret the components of the Black–Scholes–Merton model as applied to call options in terms of a leveraged position in the underlying.

The BSM model call value can be interpreted as a stock component, $SN(d_1)$, minus a bond component, $e^{-rT}XN(d_2)$. This is visible in the call option formula: $c = SN(d_1) - e^{-rT}XN(d_2)$.
An option can be thought of as a dynamically managed portfolio of the underlying stock and zero-coupon bonds, where the bonds are sold to finance the purchase of the stock.

LO: Describe how the Black–Scholes–Merton model is used to value European options on equities and currencies.

The BSM model for non-dividend paying stock call option is:
$c = SN(d_1) - e^{-rT}XN(d_2)$

The value of a put option on a non-dividend paying stock is given by:
$p = e^{-rT}XN(-d_2) - SN(-d_1)$

The value of a currency call option is given by:
$c = Se^{-r_f T}N(d_1) - e^{-rT}XN(d_2)$

The value of a currency put option is given by:
$p = e^{-rT}XN(-d_2) - Se^{-r_f T}N(-d_1)$

LO: Describe how the Black model is used to value European options on futures.

The Black model for European-style futures options is as follows:
$c = e^{-rT}[F_0(T)N(d_1) - XN(d_2)]$

$p = e^{-rT}[XN(-d_2) - F_0(T)N(-d_1)]$

LO: Describe how the Black model is used to value European interest rate options and European swaptions.

Under the Black model the interest rate call value is given by:
$c = (AP)e^{-r\,(tj-1\,+\,tm)}\,[FRA(0, t_{j-1}, t_m)\ N(d_1) - R_XN(d_2)]$

Under the Black model the interest rate put value is given by:
$p = (AP)\ e^{-r(tj-1\,+\,tm)}[R_XN(-d_2) - FRA(0, t_{j-1}, t_m))N(-d_1)]$

Under the Black model, the payer swaption valuation model for $1 notional amount is:
$PAY_{SWN} = (AP)PVA[R_{FIX}N(d_1) - R_XN(d_2)]$

Under the Black model, the receiver swaption valuation model for $1 notional amount is:
$REC_{SWN} = (AP)PVA[R_XN(-d_2) - R_{FIX}N(-d_1)]$

LO: Interpret each of the option Greeks.

Delta is defined as the change in a given portfolio for a given small change in the value of the underlying instrument, holding everything else constant.

Gamma is defined as the change in a given portfolio delta for a given small change in the value of the underlying instrument, holding everything else constant.

Theta is defined as the change in the value of an option given a small change in calendar time, holding everything else constant.

Vega is defined as the change in a given portfolio for a given small change in volatility, holding everything else constant.

Rho is defined as the change in a given portfolio for a given small change in the risk-free interest rate, holding everything else constant.

LO: Describe how a delta hedge is executed.

Delta hedging refers to managing the portfolio delta by entering additional positions into the portfolio. If Delta_H is the delta of the hedging instrument, the optimal number of units of the hedging instruments, N_H, is given by the formula below:

$$N_H = \frac{-\text{ Original Portfolio delta}}{\text{Delta}_H}$$

LO: Describe the role of gamma risk in options trading.

Stock prices in reality tend to 'jump' suddenly rather than move continuously and smoothly.

This creates a 'Gamma Risk', i.e., a risk that stock prices would jump while hedging a position and suddenly leave the position un-hedged.

LO: Define implied volatility and explain how it is used in options trading.

Implied volatility is the BSM model volatility that yields the market option price. Implied volatility is a measure of future volatility, whereas historical volatility is a measure of past volatility. Implied volatility can be used to compare the values of two options, with different exercise prices and expiration dates.

