Portfolio Management I

Learning Module 1
Portfolio Risk and Return: Part I

LOS: Describe characteristics of the major asset classes that investors consider in forming portfolios.

LOS: Explain risk aversion and its implications for portfolio selection.

LOS: Explain the selection of an optimal portfolio, given an investor's utility (or risk aversion) and the capital allocation line.

LOS: Calculate and interpret the mean, variance, and covariance (or correlation) of asset returns based on historical data.

LOS: Calculate and interpret portfolio standard deviation.

LOS: Describe the effect on a portfolio's risk of investing in assets that are less than perfectly correlated.

LOS: Describe and interpret the minimum-variance and efficient frontiers of risky assets and the global minimum-variance portfolio.

Introduction

An investor should seek to build an optimal portfolio, given specific return and risk preferences. This learning module examines the construction of efficient portfolios. We will seek to understand how return and risk preferences can set the stage for narrowing down an infinite number of potential portfolios to craft an optimal portfolio for one investor.

Two of the most important factors in creating an optimal portfolio are the particular investment characteristics of the individual assets being considered and the correlations between those assets.

LOS: Describe characteristics of the major asset classes that investors consider in forming portfolios.

Historical Return and Risk

Historical return refers to the return that was actually earned in the past, while expected return refers to the return that an investor expects to earn in the future. Historical returns are calculated from historical data, while expected returns are determined by the real risk-free interest rate, expected inflation, and expected risk.

Even though investors sometimes use historical returns to forecast expected returns, bear in mind that there is no guarantee that returns earned in the past will be produced in the future.

Risk-Return Trade-Off

Every investment decision involves a trade-off between return and risk. Empirical evidence has shown that, over the long run, market prices will reward higher risk with higher returns.

Other Investment Characteristics

In order to evaluate investments using mean (expected return) and variance (risk), we need to make the following assumptions:

- Returns follow a normal distribution, which is fully described by mean and variance.
- Markets are informationally and operationally efficient.

When these assumptions are violated, we need to consider additional investment characteristics.

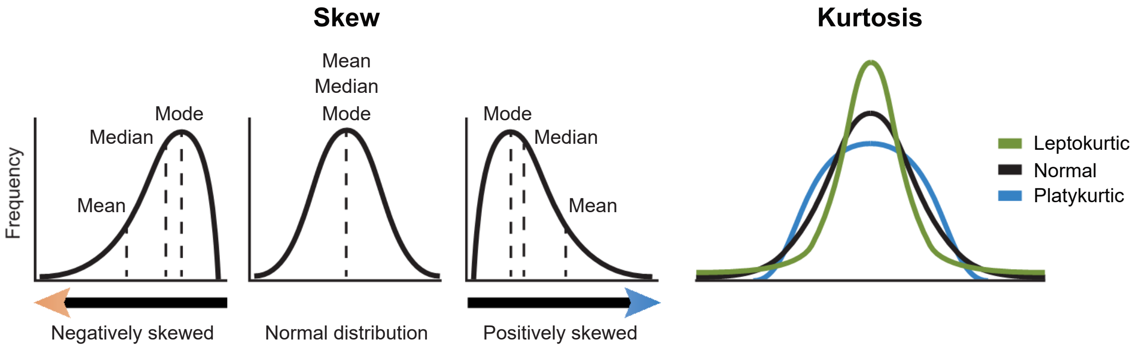
Distributional Characteristics

We have discussed the characteristics of a normal distribution in the Quantitative Methods readings. The mean-variance framework is only appropriate for evaluating investments whose returns are normally distributed. In reality, however, returns are not always normally distributed. Deviations from that norm may occur because of either skewness or kurtosis.

Skewness refers to the asymmetry of a return's distribution. When most of the distribution is concentrated on the left, it is referred to as right skewed or positively skewed. When most of the distribution is concentrated to the right, it is referred to as left skewed or negatively skewed.

Kurtosis refers to fat tails, or higher-than-normal probabilities for extreme returns. This leads to an increase in an asset's risk that is not captured by the mean-variance framework.

Exhibit 1 Skewness and kurtosis



Market Characteristics

Markets are not always operationally efficient. One limitation on operational efficiency in markets is liquidity. Liquidity has an impact on a security's bid-ask spread (eg, illiquid stocks have a wider spread) and on the price impact of a trade (eg, illiquid stocks suffer a greater price impact).

Risk Aversion and Portfolio Selection

✓ **LOS:** Explain risk aversion and its implications for portfolio selection.

Suppose an investor is offered two alternatives:

- Option 1: A guarantee of $25 in one year
- Option 2: A 50% chance of $50 in one year, and a 50% chance of nothing in one year

The *expected* value in both cases is $25, but there are three possibilities regarding the investor's preferences.

The investor may play it safe and go with Option 1. This behavior is indicative of risk aversion. Risk-averse investors aim to maximize returns for a given level of risk and minimize risk for a given level of return. Historically, there has been a positive relationship between risk and return, which suggests that market prices are primarily determined by investors who are predominantly risk averse.

The investor may choose to "gamble" and go with Option 2. Such risk-seeking investors get extra utility or satisfaction from the uncertainty of the outcome associated with their investments. Even though most individuals do exhibit risk-seeking behavior in isolated situations (eg, gambling at casinos when the expected value of the payoff is negative), individuals are assumed to be risk averse with their investments.

The investor may be indifferent between the two options. Such risk-neutral investors seek higher returns irrespective of the level of risk inherent in an investment.

Risk tolerance refers to the level of risk that an investor is willing to accept to achieve her investment goals. The lower the risk tolerance, the lower the level of risk acceptable to the investor. The lower the risk tolerance, the higher the level of risk aversion.

Utility Theory and Indifference Curves

For investment management, utility is a measure of the relative satisfaction that an investor derives from a particular portfolio. For example, a risk-averse investor obtains a higher utility from a definite (ie, riskless) outcome relative to an uncertain (ie, risky) outcome with the same expected value. In order to quantify the preferences for investment choices using risk and return, utility functions are used. An example of a utility function is:

Equation 1

$$\text{Investor utility } (U_i) = \underbrace{E(R_i)}_{\text{Expected return}} - \frac{1}{2}\overbrace{A}^{\text{Investor's risk aversion}}\underbrace{\sigma_i^2}_{\text{Expected variance}}$$

In Equation 1, *A* is a measure of an investor's risk aversion. A risk-averse investor demands additional return to accept more risk. The utility function assumes that:

- investors are *generally* risk averse but prefer more return to less return,
- investors are able to rank different portfolios based on their preferences, and
- these preferences are internally consistent.

This means that if Investment 1 is preferred to Investment 2, and Investment 2 is preferred to Investment 3, Investment 1 *must* be preferred to Investment 3.

We can draw the following conclusions from this utility function: utility is unbounded on both sides, and it can be highly negative or highly positive. Greater return results in higher utility. Greater risk results in lower utility. The greater the value of *A*, the greater the negative effect of risk on utility.

Utility cannot be compared across individuals because it is a personal concept. Consequently, it cannot be summed among individuals to determine utility from a group or societal standpoint.

Note that utility is not an absolute level of satisfaction. A portfolio with a utility, *U*, of 2.5 would be preferred to one with a *U* of 1.25, but that does mean that the higher utility portfolio is twice as satisfying as the lower one.

The risk-free asset, as it has a variance (risk) of zero, generates the same utility for all investors.

Points to remember about the risk aversion coefficient *A*:

A is positive for a risk-averse investor: additional risk reduces total utility.

A is zero for a risk-neutral investor: additional risk has no impact on total utility.

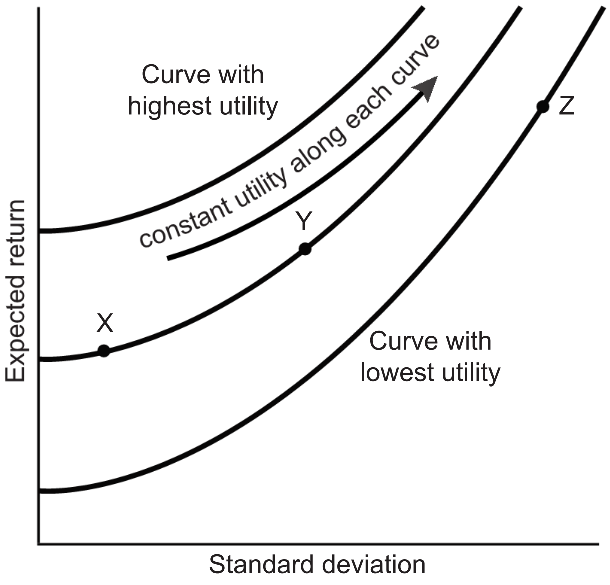
A is negative for a risk-seeking investor: additional risk enhances total utility.

Indifference Curves

The risk-return trade-off that an investor is willing to bear can be illustrated by an **indifference curve**. An investor realizes the same total utility or satisfaction from every point on a given indifference curve. Since each investor can have an infinite number of risk-return combinations that generate the same utility, indifference curves are continuous at all points.

Indifference curves are **upward sloping**. This means that an investor will be indifferent between two investments with different expected returns only if the investment with the lower expected return entails a lower level of risk as well. Indifference curves are also **curved**, and their slope becomes steeper as more risk is taken. The return required for every unit of additional risk grows at an increasing rate because of the diminishing marginal utility of wealth.

Exhibit 2 Indifference curves for a single investor

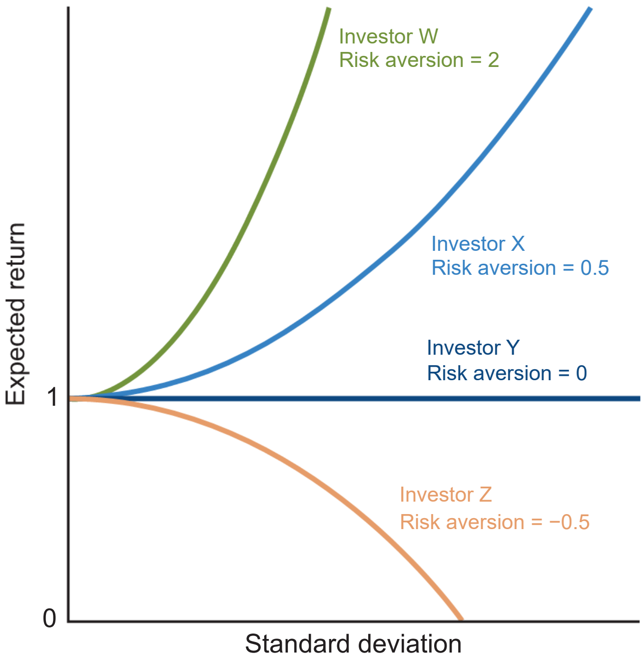


The investor's set of indifference curves includes plots of the utility function. A given investor has:

- the same utility for points on the same curve,
- different utilities for points on different curves, and
- greater utility for points on a higher curve.

A higher curve (ie, more to the "northwest" of the graph) represents greater utility than a lower curve (ie, more to the "southeast") since it represents more return for the same risk at a corresponding point on the lower curve. In Exhibit 2, Investment Y has a greater utility for the investor than Investment Z since Investment Y is on a higher curve than Investment Z. Investment Z has lower utility than Investment X since Investment Z is on a lower indifference curve, although it has a higher rate of return (and is therefore higher on the graph).

Exhibit 3 Indifference curves for various types of investors



The slope of an indifference curve represents the extra return required by the investor to accept an additional unit of risk.

As illustrated in Exhibit 3, a risk-averse investor has a relatively steep indifference curve (significant extra return required to take on more risk). A less risk-averse investor has a flatter indifference curve (lower extra return required to take on more risk). A risk-neutral investor would have a perfectly horizontal indifference curve, as his utility does not vary with risk. A risk-seeking investor would have an indifference curve with a negative slope as her utility increases with higher return and higher risk.

Remember, the risk aversion coefficient (in the utility function) and the slope of the indifference curve are positively related.

For the remainder of this reading, we will assume that all investors are risk averse, unless noted.

Example 1 Calculation of utility

The questions below are based on the following information:

Investment	Expected return $E(r)$	Standard deviation (σ)
A	8%	19%
B	10%	24%
C	17%	28%
D	24%	32%

Calculate

Using the utility formula: $U = E(r) - 1/2\ A\sigma^2$

1. Which investment will a risk-averse investor with a risk aversion coefficient of 5 choose?
2. Which investment will a risk-averse investor with a risk aversion coefficient of 3 choose?
3. Which investment will a risk-neutral investor choose?
4. Which investment will a risk-loving (seeking) investor choose?

Solution

This table shows the utility for risk-averse investors with $A = 5$ and $A = 3$.

Investment	Expected return $E(r)$	Standard deviation (σ)	Utility at $A = 5$	Utility at $A = 3$
A	8%	19%	−0.010	0.0256
B	10%	24%	−0.044	0.0136
C	17%	28%	−0.026	0.0524
D	24%	32%	−0.016	0.0864

1. A risk-averse investor with a risk aversion coefficient of 5 would choose Investment A as it has the highest utility for that investor.
2. A risk-averse investor with a risk aversion coefficient of 3 would choose Investment D as it has the highest utility for that investor.
3. A risk-neutral investor's risk aversion coefficient is 0. She wants the highest return possible. Therefore, she would choose Investment D.
4. A risk-loving investor does not mind seeking higher risk when reaching for a higher return. He would choose investment D as well.

Application of Utility Theory to Portfolio Selection

LOS: Explain risk aversion and its implications for portfolio selection.

LOS: Explain the selection of an optimal portfolio, given an investor's utility (or risk aversion) and the capital allocation line.

The expected return on a risk-free asset is entirely certain, and therefore the standard deviation of its expected returns is zero ($\sigma_{Rf} = 0$). The return earned on the risk-free asset is the risk-free rate, R_f.

Portfolio Containing a Risky Asset and the Risk-Free Asset

Let's assume that we invest a proportion (w_1) of our investable funds in a risk-free asset that has an expected return of R_f and a variance of zero, and ($1 - w_1$) in a risky asset that has an expected return of $E(R_i)$ and variance of $\sigma_i{}^2$.

The *expected return* for the portfolio, $E(R_p)$, that includes the risk-free asset and risky asset is simply the weighted average of their expected returns.

Equation 2a

$$E(R_p) = w_1 R_f + (1 - w_1)E(R_i)$$

Equation 2b

$$\sigma_p^2 = w_1^2\sigma_f^2 + (1 - w_1)^2\sigma_i^2 + 2w_1(1 - w_1)p_{1,2}\sigma_f\sigma_i$$

$$\sigma_p^2 = (1 - w_1)^2\sigma_i^2$$

Equation 2c

$$\sigma_p^2 = (1 - w_1)\sigma_i$$

The variance and standard deviation of the risk-free asset are zero because it has a guaranteed return. Further, since the return on the risk-free asset does not vary with the return on any risky asset, the correlation between the risk-free asset and the risky asset is also zero. Then, the portfolio variance (in Equation 2b) can be simplified as shown. The portfolio standard deviation (shown in Equation 2c) is derived by taking the square root of each side (of Equation 2b).

The expression for the standard deviation of the portfolio (Equation 2c) can be reorganized to obtain an expression for w_1:

$$w_1 = 1 - \frac{\sigma_p}{\sigma_i}$$

Substituting terms, we can state the expected return on the portfolio as a function of portfolio risk:

$$E(R_p) = \left(1 - \frac{\sigma_p}{\sigma_i}\right) R_f + \frac{\sigma_p}{\sigma_i} E(R_i)$$

And then:

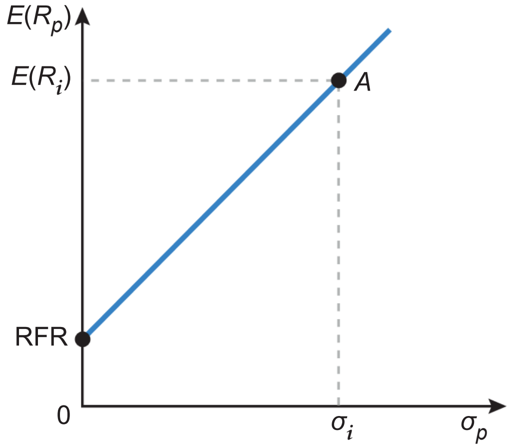
$$E(R_p) = R_f + \frac{(E(R_i) - R_f)}{\sigma_i} \sigma_p$$

This equation, which relates the return on the portfolio composed of the risk-free asset and risky asset to the standard deviation of the portfolio, is known as the **capital allocation line** (CAL). The CAL has an intercept of R_f and a constant slope that equals:

$$\frac{(E(R_i) - R_f)}{\sigma_i}$$

The expression for the slope of the CAL is the extra return required for each additional unit of risk; this is also known as the market price of risk.

Exhibit 4 Capital allocation line with two assets



At point R_f, the portfolio consists *only* of *risk-free* assets. Its expected return is R_f and its variance equals zero.

At point *A*, the portfolio consists *only* of *risky* assets. Its expected return equals $E(R_i)$ and its variance equals σ_i^2.

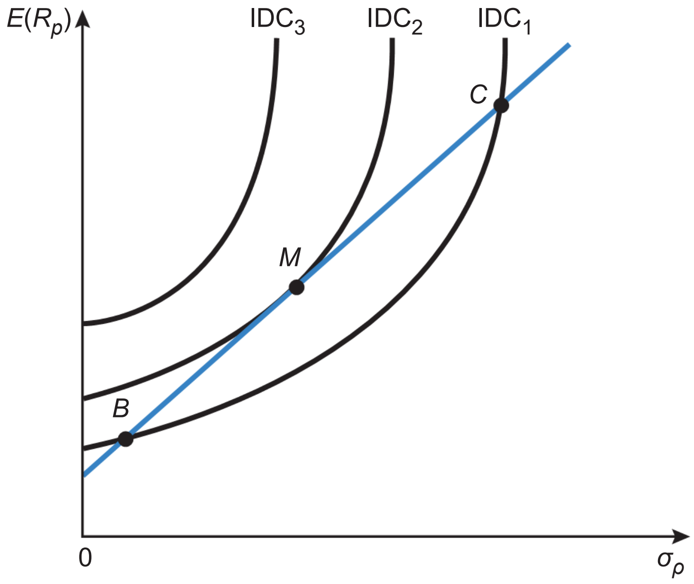
Risk-return combinations beyond point A can only be obtained by *borrowing* at the risk-free rate and *investing* in the risky asset. By doing so, the investor creates a portfolio where the weight of the risky asset is *greater* than 100% and the weight of the risk-free asset is *less* than 0%. Note, however, that adding the two weights will still result in a portfolio with a weight of 100%.

The next question is which of the numerous portfolios that lie along the CAL will actually be chosen by the investor. The answer lies in combining indifference curves with the CAL. Indifference curves represent the investor's utility function, while the CAL represents the risk-return combinations of the set of portfolios that the investor can invest in.

- Investors can invest in portfolios that lie below the CAL, but the investor would not be maximizing the potential return given the level of risk she is willing to take.
- Portfolios that lie above the CAL are desirable but cannot be attained with the given assets.

Exhibit 5 demonstrates this concept:

Exhibit 5 Portfolio selection



In Exhibit 5, notice that:

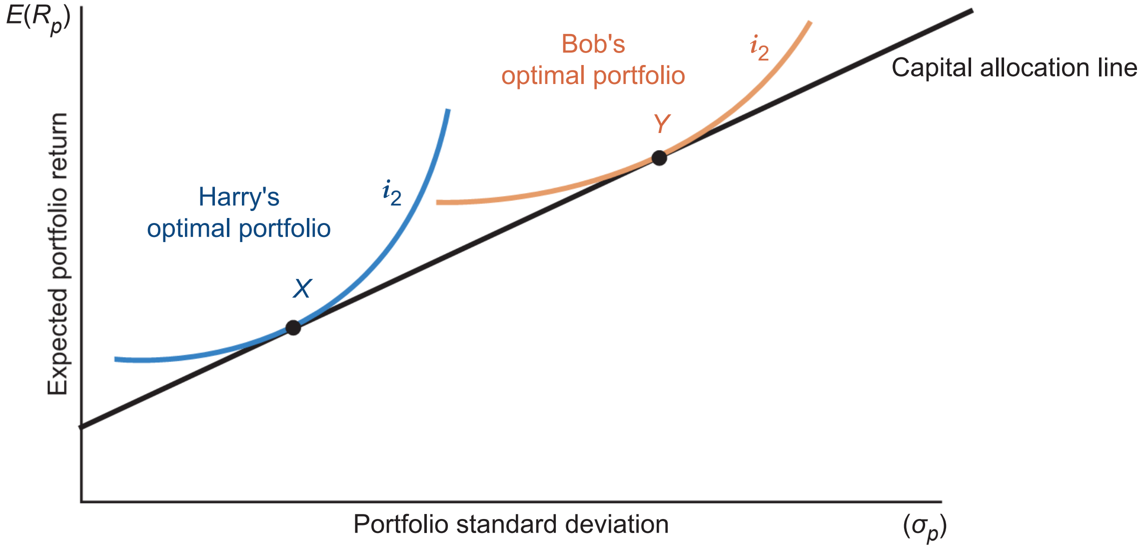
- IDC_3, which lies above the CAL, is the most desirable but cannot be attained with the available assets.
- IDC_1 intersects the CAL at two different points, point B and point C. Both these points offer the same level of satisfaction to the investor, as they lie on the same indifference curve.
- IDC_2 is tangential to the CAL at point M.

Given a choice between investing in a portfolio on IDC_1 (points B or C) or in a portfolio on IDC_2 (point M) an investor would choose the portfolio on IDC_2 as it offers a higher level of satisfaction (it lies to the northwest of IDC_1). Point M, known as the point of tangency between the investor's indifference curve and the capital allocation line, represents the optimal portfolio for this investor.

Portfolio Selection for Two Investors with Differing Levels of Risk Aversion

Exhibit 6 shows the indifference curves for Harry and Bob. We see that the indifference curve of a relatively more risk-averse investor (ie, Harry) lies to the left of the indifference curve of a less risk-averse investor (ie, Bob) because Harry has a lower tolerance for risk. Notice that Harry's optimal portfolio has a lower expected return and a lower level of risk than Bob's optimal portfolio. For the same level of return, Harry's indifference curve has a steeper slope, which suggests that he would need to have a greater incremental return than Bob for taking on additional risk.

Exhibit 6 Optimal portfolios and risk aversion



Portfolio Risk and Portfolio of Two Risky Assets

LOS: Calculate and interpret portfolio standard deviation.

LOS: Describe the effect on a portfolio's risk of investing in assets that are less than perfectly correlated.

Earlier we learned that in addition to being a function of individual asset variances and their weights in the portfolio, a portfolio's variance also depends on the covariance and the correlation between the assets in the portfolio. The formula for the standard deviation of a portfolio of risky assets is:

$$\sigma_p = \sqrt{\sigma_p^2} = \sqrt{\sum_{i=1}^{n} w_i^2 \sigma_u^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \text{Cov}_{i,j}} \quad i \neq j$$

The formula for the standard deviation of a portfolio consisting of *two* risky assets is:

$$\sigma_p = \sqrt{w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1 w_2 \sigma_1 \sigma_2 p_{1,2}}$$

And since $\sigma_1 \sigma_2 \rho_{1,2}$ = Covariance, the formula can also be written as:

$$\sigma_p = \sqrt{w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1 w_2 \text{Cov}_{1,2}}$$

The first part of the formula for a two-asset portfolio standard deviation ($w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2$) tells us that portfolio standard deviation is a positive function of the standard deviation and weights of the individual assets held in the portfolio.

The second part ($2w_1w_2\text{Cov}_{1,2}$) shows us that portfolio standard deviation also depends on how the two assets move in relation to each other (eg, covariance or correlation).

The maximum value for portfolio standard deviation will be obtained when the correlation coefficient equals +1 (perfect positive correlation). In that case, there are no diversification benefits.

When asset returns are positively correlated, the second part of the formula for portfolio standard deviation is also positive, and portfolio standard deviation is higher than when the correlation coefficient equals zero. If the correlation coefficient equals zero, the second part of the formula will equal zero and portfolio standard deviation will fall somewhere in between.

When asset returns are negatively correlated, the final term in the standard deviation formula is negative and serves to reduce portfolio standard deviation. The portfolio standard deviation will be minimized when the correlation coefficient equals −1.

The formula for the standard deviation of a portfolio consisting of three risky assets is:

$$\sigma_p = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2 + 2w_1w_2\text{Cov}_{1,2} + 2w_2w_3\text{Cov}_{2,3} + 2w_3w_1\text{Cov}_{3,1}}$$

Let’s go through an example to highlight some important points:

Example 2 Correlations

Asset	***E*(*R*)**	**Weight**	**Variance**	**Std. dev.**
A	0.10	0.50	0.0081	0.0900
B	0.15	0.50	0.0049	0.0700

Portfolio standard deviations in various correlation scenarios are given below:

Scenario	**Correlation between assets**	**Portfolio standard deviation**
a	−1.0	0.0100
b	−0.5	0.0409
c	0.0	0.0570
d	+0.5	0.0695
e	+1.0	0.0800

When viewing this example, remember that the portfolio’s expected return *does not* vary with the correlation coefficient of the two assets. The expected return for this portfolio equals 0.075 or 7.50%.

With perfect negative correlation, the standard deviation of the portfolio is at its lowest (1.00%). The negative covariance term significantly offsets the individual asset variance terms.

With a correlation of −0.5, the standard deviation of the portfolio is not at its lowest but is still relatively low (4.09%) due to the negative covariance term.

With zero correlation, the portfolio standard deviation (5.70%) is higher than it is with negative correlation, but lower than it is with positive correlation.

With a correlation of +0.5, the standard deviation of the portfolio is still higher, at 6.95%.

With perfect positive correlation, the standard deviation of the portfolio (8.00%) is at its highest possible level. There are no diversification benefits from investing in the portfolio when correlation is +1. In this scenario, the portfolio standard deviation is simply the weighted average of the standard deviations of the individual assets. This is because the expression for portfolio standard deviation is in the form of:

$$\sigma_{\text{port}} = \sqrt{(w_A\sigma_A)^2 + (w_B\sigma_B)^2 + 2(w_A\sigma_A)(w_B\sigma_B)(1)}$$

Which can be simplified to:

$$\sigma_{\text{port}} = \sqrt{[(w_A\sigma_A) + (w_B\sigma_B)]^2}$$

$$\sigma_{\text{port}} = (w_A\sigma_A) + (w_B\sigma_B)$$

Recall from algebra that an expression in the form of $(a + b)^2$ can be expanded to $(a^2 + b^2 + 2ab)$.

In conclusion, the risk (standard deviation) of a portfolio of risky assets depends on the asset weights and standard deviations and, *most importantly*, on the correlation of asset returns. The *greater* the correlation between the individual assets, the *greater* the portfolio's standard deviation.

Constant Correlation with Changing Weights

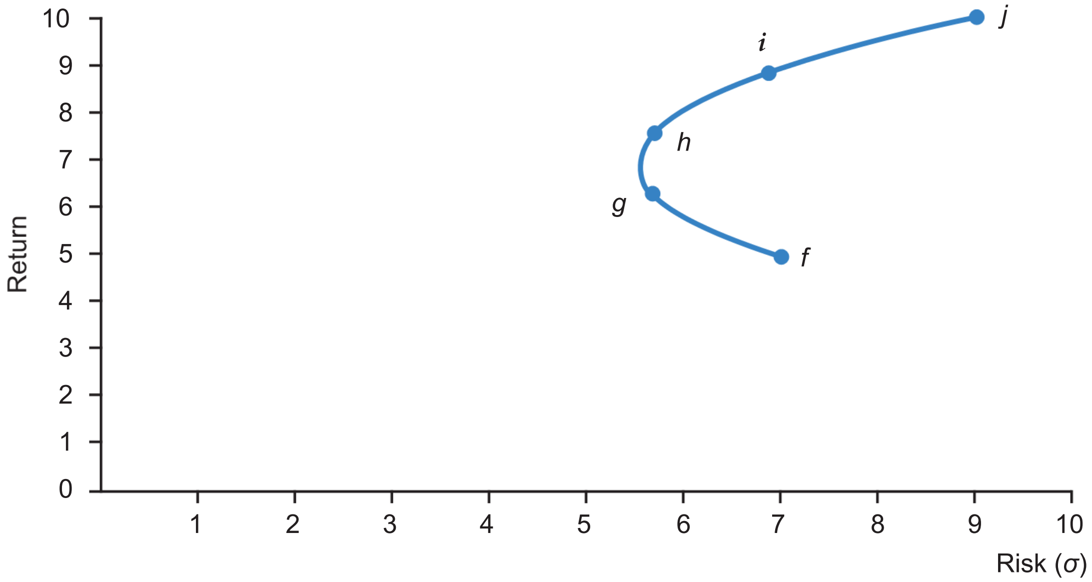
Using the same two assets, now let's change the weights of the individual assets in the portfolio and use a constant correlation coefficient of zero to gauge the impact on portfolio standard deviation:

Example 3 Correlations, part 2

Portfolio	W_A	W_B	$E(R_p)$	Portfolio std. dev.
f	0	0.50	0.0500	0.0700
g	0.25	0.75	0.0625	0.0500
h	0.75	0.50	0.0750	0.0500
i	0.75	0.25	0.0875	0.0700
j	1.00	0	0.1000	0.0900

Exhibit 7 shows the graph of the return and standard deviation for each portfolio:

Exhibit 7 Risk-return trade-off



In Exhibit 7, notice the following:

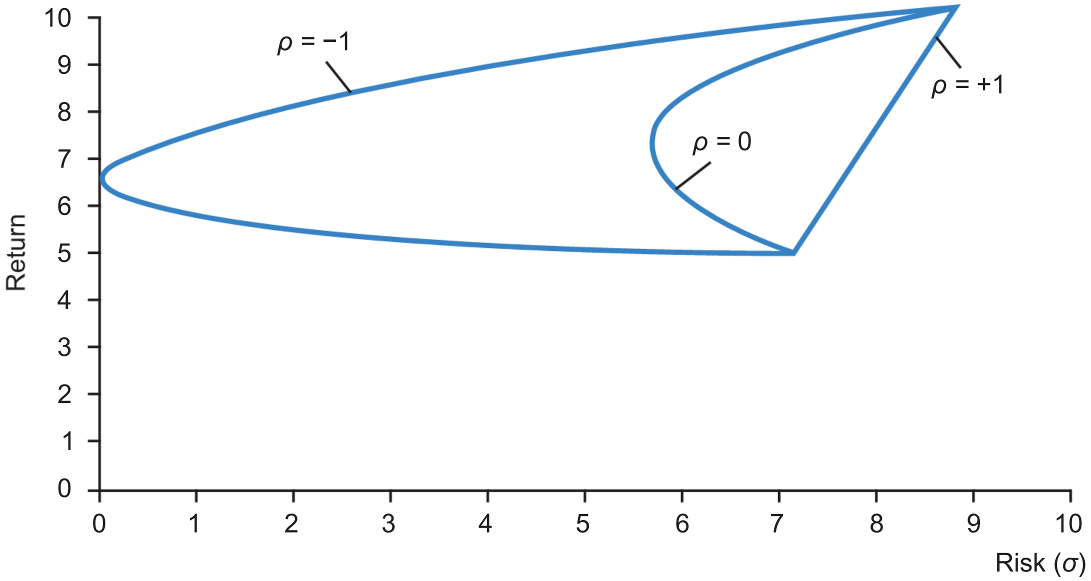
The set of risk-return combinations for the five portfolios traces an ellipse that starts at Portfolio f (Asset B's risk-return profile) and goes through the 50-50 point to Portfolio j (Asset A's risk-return profile).

With assets that have a correlation of less than +1, it is possible to form portfolios that have less risk than either of the individual assets. This holds true for Portfolios g, h, and i in our example.

A conservative investor can experience both a greater return and less risk by diversifying into a greater-risk, greater-return asset if the correlation between the assets is fairly low. Suppose a conservative investor is fully invested in Asset B (Portfolio f), where she expects a return of 5.00% with a portfolio standard deviation of 7.00%. By shifting 25.00% of her assets into the greater-risk asset (Asset A) and investing in Portfolio g, she can increase her expected return to 6.25% and lower the standard deviation of her portfolio to 5.71%.

Exhibit 8 shows the change in curvature of the risk-return relationship between assets depending on their weights, as a function of the correlation between the two assets.

Exhibit 8 Effect of correlation on portfolio risk and return



Notice that for any mix of asset weights, the maximum value of the portfolio standard deviation can only be equal to the largest of the individual assets' standard deviations.

In Exhibit 8, notice the following:

When correlation equals +1, the risk-return combinations that result from altering the weights lie along a straight line between the two assets' risk-return profiles.

As correlation falls, the curvature of this line increases.

When correlation equals (−1), the curve becomes tangent to the vertical axis. This tangency point represents a zero-risk portfolio where portfolio return must equal the risk-free rate to prevent arbitrage.

Portfolio of Many Risky Assets

As more and more assets are added to a portfolio, the contribution of each individual asset's risk to portfolio risk diminishes. The covariance among the assets in the portfolio accounts for the bulk of portfolio risk.

Further, given that there are a large number of assets in the portfolio, if we assume that all assets in the portfolio have the same variance and the same correlation among assets, the portfolio can have a variance of zero (ie, zero risk) if the individual assets are unrelated to one another.

The Power of Diversification

Investors can diversify by investing in a variety of asset classes (eg, large-cap stocks, small-cap stocks, bonds, commodities, real estate) that are not typically highly correlated. There are various ways investors can diversify:

- Index funds minimize the costs of diversification and grant exposure to specific asset classes.
- Investing among countries that focus on different industries, are undergoing different stages of the business cycle, and have different currencies. Since stock returns are often not correlated with currency returns, currency diversification is an important benefit of international investing.
- Choosing not to invest a significant portion of their wealth in employee stock plans as their human capital is already entirely invested in their employing companies.
- Only adding a security to the portfolio if its Sharpe ratio is greater than the product of the Sharpe ratio of the portfolio and the correlation coefficient.
- Only adding a security to the portfolio if the benefit (additional expected return, reduced portfolio risk) is greater than the associated costs (trading costs and costs of tracking a larger portfolio).
- Adding insurance to the portfolio by purchasing put options or adding an asset class that has a negative correlation with the assets in the portfolio (eg, commodities).

Efficient Frontier: Investment Opportunity Set and Minimum Variance Portfolios

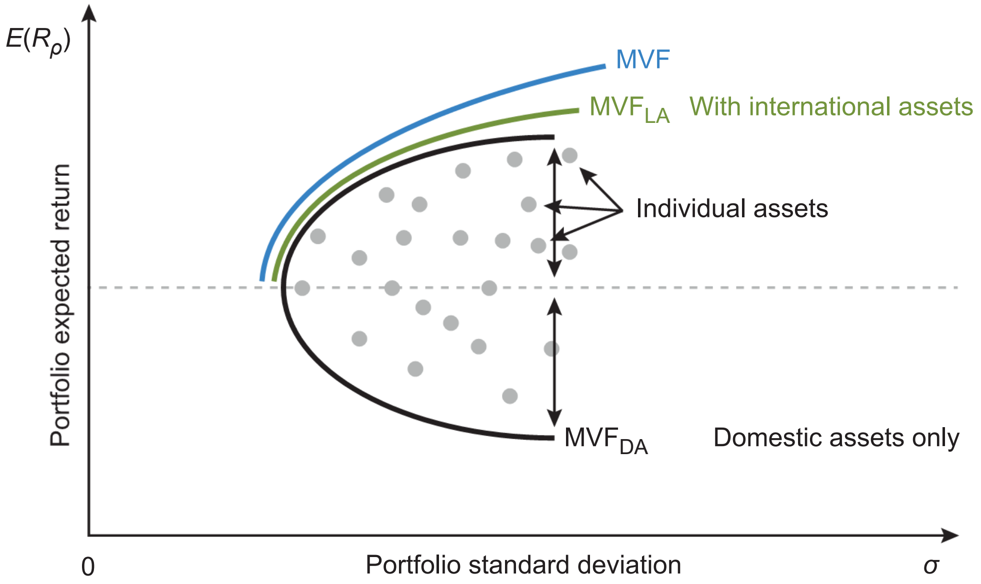
LOS: Describe the effect on a portfolio's risk of investing in assets that are less than perfectly correlated.

LOS: Describe and interpret the minimum-variance and efficient frontiers of risky assets and the global minimum-variance portfolio.

We have learned that combining risky assets may result in a portfolio that has lower risk than any of the individual assets in the portfolio. As the number of available assets increases, they can be combined into various portfolios (each with different assets and weights) to create an opportunity set of investments.

Combinations of these assets can be formed into portfolios that have the lowest level of risk for each level of expected return. An envelope curve that plots the risk-return characteristics of the lowest-risk "domestic assets only" portfolios is labeled MVFDA (minimum-variance frontier—domestic assets), seen in Exhibit 9.

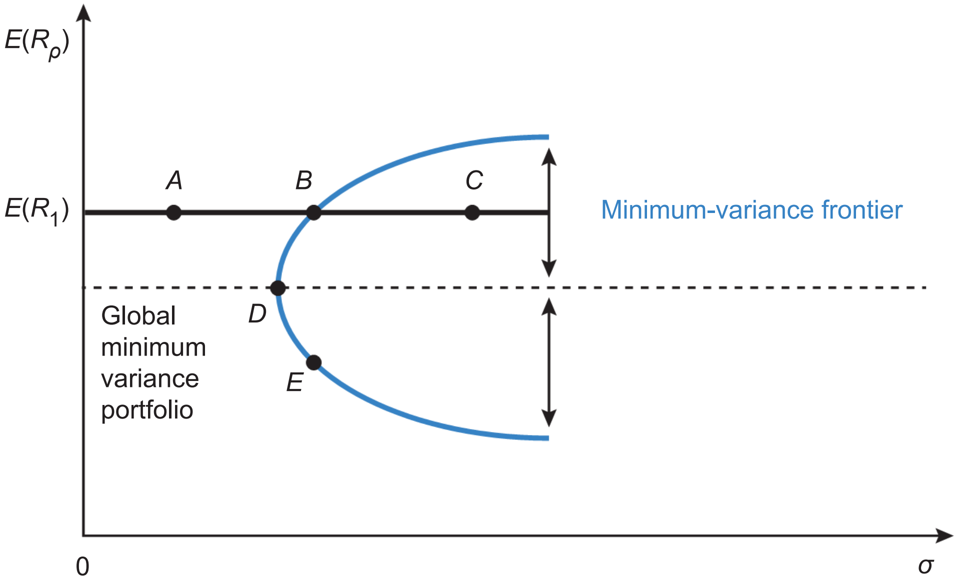
Exhibit 9 Investment opportunity set



As international assets are added to the portfolio, portfolio risk for each level of return can be reduced further, given that international assets are not perfectly positively correlated with domestic assets. Thus, the minimum-variance frontier that includes international assets (MVFIA) lies to the left of MVFDA. Similarly, once all possible investments and asset classes are considered, the minimum-variance frontier (MVF) that plots the risk-return characteristics of portfolios that minimize portfolio risk at each given level of return lies further to the left.

As we move forward, we will work with this minimum-variance frontier and assume that all assets and asset classes have been considered in deriving the frontier. It is also assumed that no risk-averse investor would invest in any portfolio that lies to the right of the MVF as it would mean a higher level of risk than a portfolio that lies on the MVF for a given level of return.

Exhibit 10 Minimum-variance frontier



In Exhibit 10, Portfolios A, B, and C all have the same expected return, $E(R_1)$. Portfolio A has the lowest level of risk but lies to the left of the minimum variance frontier, so it is unattainable given the investment opportunity set. The minimum risk that the investor can take to earn $E(R_1)$ is investing in Portfolio B. A risk-averse investor would invest in Portfolio B over Portfolio C as it entails a lower risk for the same expected return. Portfolio D defines the global minimum-variance portfolio. It is the portfolio of risky securities that entails the lowest level of risk among all the risky asset portfolios on the minimum-variance frontier.

Note that the emphasis here is on risky assets. Later, the introduction of the risk-free asset will allow us to relax this constraint.

The minimum-variance frontier represents portfolios with the lowest level of risk for each level of expected return. Investors aim to maximize the expected return for each level of risk. In Exhibit 10, given a choice between Portfolios B and E, which both lie on the minimum-variance frontier and have the same level of risk, an investor would prefer Portfolio B as it offers a higher return. Therefore, all portfolios on the MVF that lie above and to the right of the global minimum-variance portfolio dominate all portfolios on the MVF that lie below and to the right of the global minimum-variance portfolio.

This dominant portion of the MVF is known as the Markowitz efficient frontier. It contains all the possible portfolios that rational, risk-averse investors will consider investing in.

An important thing to note about the efficient frontier is that the slope of the curve *decreases* as an investor moves to the right. The additional return attained as investors take on more risk *declines*.

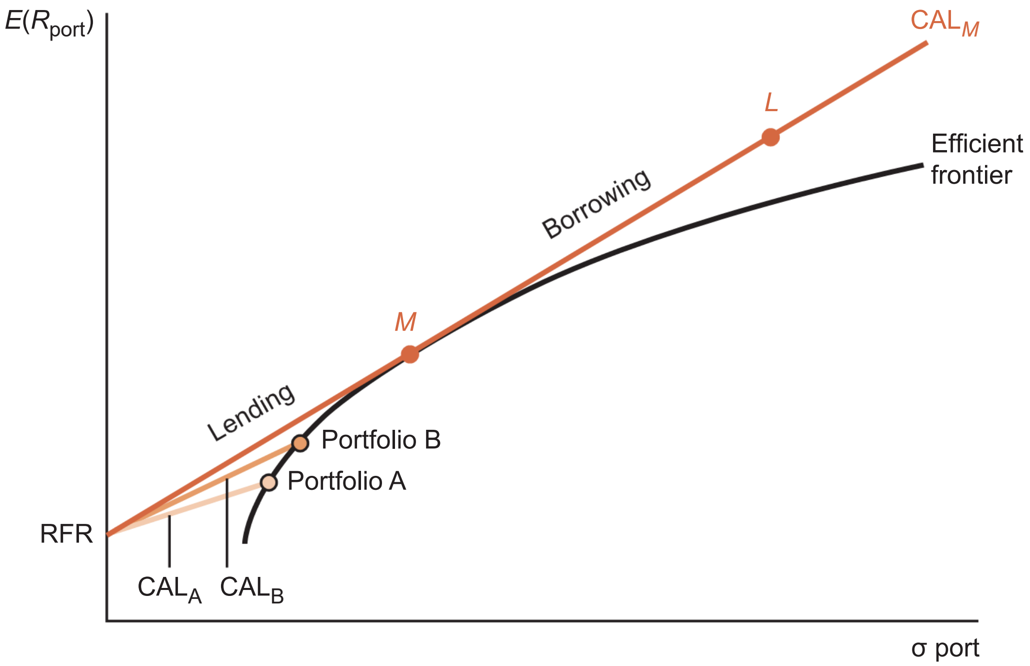
Efficient Frontier: A Risk-Free Asset and Many Risky Assets

✓ **LOS:** Explain the selection of an optimal portfolio, given an investor's utility (or risk aversion) and the capital allocation line.

Capital Allocation Line and Optimal Risky Portfolios

Now let's include the risk-free asset in our analysis. The risk-free asset has zero risk (so it plots on the *y*-axis), an expected return of the risk-free rate (R_f), and an assumed zero correlation with risky assets. Further, the risk-return characteristics of portfolios that combine the risk-free asset with a risky asset or a portfolio of risky assets lie along a straight line.

Exhibit 11 Optimal risky portfolio



As Exhibit 11 demonstrates, an investor can attain any point along the capital allocation line, CAL_A, by investing a certain portion of her funds in the risk-free asset and the remainder in a portfolio of risky assets (Portfolio A). The set of portfolios that lies on CAL_A (combinations of the risk-free asset and Portfolio A with varying weights) dominates all the risky-asset portfolios on the efficient frontier below point A because portfolios along CAL_A have a higher return than the portfolios on the efficient frontier with the same risk (ie, standard deviation).

Likewise, any position can be attained along CAL_B by investing in some combination of the risk-free asset and Portfolio B of risky assets. Again, these potential combinations of the risk-free asset and Portfolio B dominate all portfolio possibilities on the efficient frontier below point B. Further, any portfolio that lies on CAL_B dominates any portfolio on CAL_A because portfolios that lie on CAL_B offer a higher expected return for any given level of risk compared with those on CAL_A.

The difference between the capital allocation lines used here and the line described earlier is that previously we were combining the risk-free asset with a risky asset. Now we are working with the Markowitz efficient frontier and are combining the risk-free asset with different portfolios of risky assets.

As the investor combines the risk-free asset with portfolios further up the efficient frontier, she attains better and better portfolio combinations. Each successive portfolio on the efficient frontier has a steeper line (ie, higher slope) linking it to the risk-free asset. The slope of this line represents the additional return per unit of extra risk. The steeper the slope of the line, the better the risk-return trade-off the portfolio offers. In Exhibit 11, the line with the steepest slope is the one drawn from the risk-free asset to Portfolio M (which occurs at the point of tangency between the efficient frontier and a straight line drawn from the risk-free rate). This particular line offers the best risk-return trade-off to the investor. Any combination of the risk-free asset and Portfolio M surpasses all portfolios below CAL_M.

It is essential to understand what the result is here. By adding the risk-free asset, we have narrowed down the choice of risky asset portfolios to a single optimal portfolio, which is at point M. The portfolio at point M is at the point of tangency between CAL_M and the efficient frontier.

At point (R_f), an investor has all her funds invested in the risk-free asset. At point M she has all of her funds invested in Portfolio M (which is entirely composed of risky securities).

At any point between (R_f) and M, she holds both Portfolio M and the risk-free asset (ie, she is *lending* some of her funds at the risk-free rate).

However, an investor may want to attain a higher expected return than available at point M. Adding *leverage* to the portfolio by *borrowing* money at the risk-free rate (R_f) and investing it in the risky asset portfolio will allow her to attain risk-return profiles beyond (to the right of, or above) point M on the CAL_M (eg, point L).

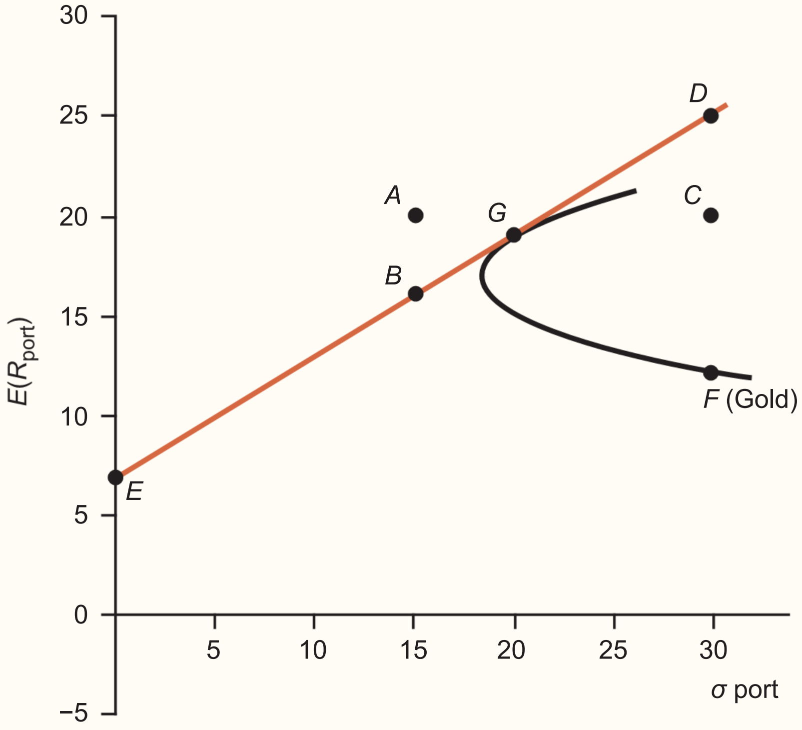
The Two-Fund Separation Theorem

The two-fund separation theorem states that regardless of risk and return preferences, all investors hold some combination of the risk-free asset and an optimal portfolio of risky assets. Therefore, the investment problem can be broken down into two steps:

1. The investing decision: The investor identifies the optimal risky portfolio.
2. The financing decision: The investor determines where exactly on the optimal CAL the portfolio will lie. The investor's risk preferences (as delineated by indifference curves) determine whether the desired portfolio requires borrowing or lending at the risk-free rate.

Example 4 Choosing the right portfolio

Based on the information in the exhibit and table, answer the following questions:



Point	Return (%)	Risk (%)
A	20	15
B	16	15
C	20	30
D	25	30
E	7	0
F (Gold)	12	30
G	19	20

1. Which of the points above is not achievable?
2. Which of the portfolios will not be chosen by a rational, risk-averse investor?
3. Which of these portfolios is most suitable for a risk-neutral investor?
4. Why is gold held by many rational investors as part of a larger portfolio, when it is shown in the graph to lie on the inefficient part of the feasible set?

Solution

1. Portfolio A lies outside the feasible set and thus it is not achievable.
2. Portfolios C and F will not be chosen by a rational, risk-averse investor.
   a. This is because Portfolio D provides a higher return (25%) than both C and F at the same level of risk (30%).
   b. Portfolios C and F are the only investable points that do not lie on the capital allocation line.
3. Portfolio D is most suitable for a risk-neutral investor who does not care about risk and wants the highest possible return.
4. Although gold lies on the inefficient part of the feasible investment opportunity set, it is still held by many rational investors as part of a larger portfolio. This is because gold, depending on the time frame, has a low or negative correlation with other traditional risky assets. This low correlation helps to reduce the overall risk of the portfolio.

Efficient Frontier: Optimal Investor Portfolio

LOS: Explain the selection of an optimal portfolio, given an investor's utility (or risk aversion) and the capital allocation line.

Investor Preferences and Optimal Portfolios

The line CAL_M in Exhibits 11 and 12 represents the best portfolios available to an investor. The portfolios along this line contain the risk-free asset and the optimal portfolio, Portfolio M, with varying weights. An individual's optimal portfolio depends on her risk-return preferences, which are incorporated into her indifference curves.

Exhibit 12 Optimal investor portfolio

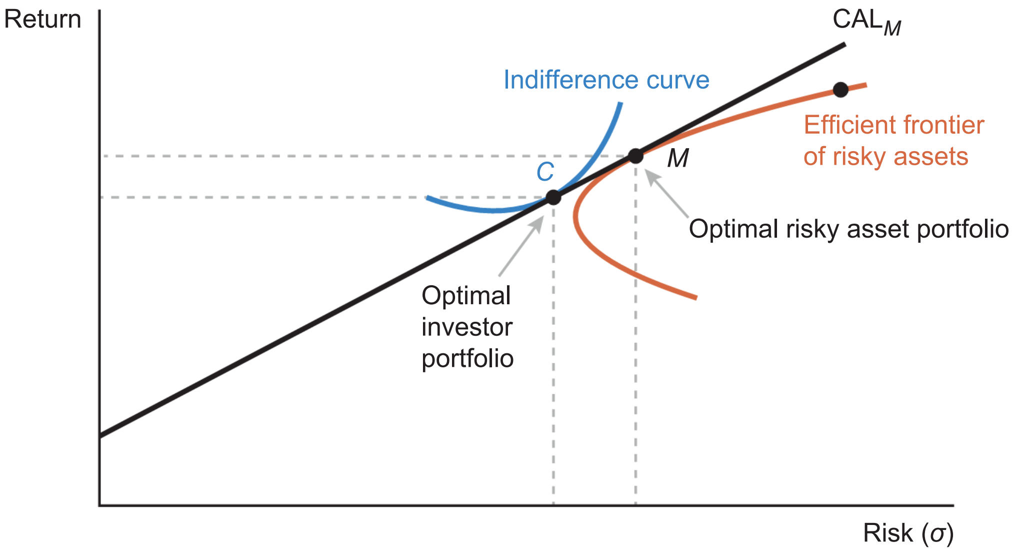


Exhibit 12 shows an investor's indifference curve, which is tangent to the CAL_M at point C. Thus, the optimal investor portfolio for this particular investor is Portfolio C on the CAL_M.

For a more risk-averse investor, the optimal investor portfolio would lie closer to the *y*-axis (a higher proportion invested in the risk-free asset), while a less risk-averse investor's optimal portfolio would lie closer to Portfolio M, and further away from the *y*-axis. An investor with an even higher tolerance for risk might borrow money at the risk-free rate to invest in Portfolio M. Her optimal portfolio would lie to the right of Portfolio M on CAL_M.

Notice that we have been able to account for all types of investors' risk preferences by using just two items—the risk-free asset and Portfolio M that consists of risky assets. Portfolio M is the optimal risky asset portfolio and will be selected by a rational, risk-averse investor regardless of her preferences. The only decision that the investor makes is how to divide her funds between the risk-free asset and Portfolio M.

Learning Module 2

Portfolio Risk and Return: Part II

✓

LOS: Describe the implications of combining a risk-free asset with a portfolio of risky assets.

LOS: Explain the capital allocation line (CAL) and the capital market line (CML).

LOS: Explain systematic and nonsystematic risk, including why an investor should not expect to receive additional return for bearing nonsystematic risk.

LOS: Explain return-generating models (including the market model) and their uses.

LOS: Calculate and interpret beta.

LOS: Explain the capital asset pricing model (CAPM), including its assumptions, and the security market line (SML).

LOS: Calculate and interpret the expected return of an asset using CAPM.

LOS: Describe and demonstrate applications of CAPM and the SML.

LOS: Calculate and interpret the Sharpe ratio, Treynor ratio, M, and Jensen's alpha.

Introduction

This learning module will help you identify the optimal risky portfolio for all investors by using the capital asset pricing model (CAPM). The basis of this reading is the computation of a portfolio's risk and return and the role correlation plays in diversifying portfolio risk and arriving at the efficient frontier.

We will review the capital market line (CML), a special case of the capital allocation line (CAL) used for passive portfolios. We will distinguish between systematic and nonsystematic risk and explain why investors are compensated for bearing systematic risk but not nonsystematic risk.

CAPM is a simple model for estimating asset returns based only on the asset's systematic risk. It allows for security selection that builds an investor's optimal portfolio by altering the asset mix beyond that of a passive market portfolio.

Before getting into the content of this learning module, let's review some of the important takeaways from the previous sections on portfolio management:

- Risky assets can be combined into portfolios that may have a lower risk than each of the individual assets in the portfolio if the assets are not perfectly positively correlated.
- An investor's investment opportunity set includes all the individual risky assets and risky asset portfolios that she can invest in. The minimum-variance frontier reduces the investment opportunity set to a curve containing only those portfolios that entail the least risk for each level of expected return.
- The global minimum-variance portfolio is the portfolio of risky assets that entails the least risk among all portfolios on the minimum-variance frontier. Investors aim to maximize return for every level of risk.

Capital Market Theory: Risk-Free and Risky Assets

✓ **LOS:** Describe the implications of combining a risk-free asset with a portfolio of risky assets.

A risk-free asset's expected return is the risk-free rate (R_f), its standard deviation is zero, and its correlation with any risky asset is zero. Once the risk-free asset is introduced into the mix:

- Any portfolio that combines a risky asset portfolio lying on the Markowitz efficient frontier and the risk-free asset has a risk-return tradeoff that is linear. Thus, the capital allocation line (CAL) is a straight line.
- A line that is drawn from the risk-free asset and is tangent to the Markowitz efficient frontier defines the optimal risky asset portfolio. This line is known as the optimal CAL.
- An investor chooses the specific optimal portfolio that contains some combination of the risk-free asset and the optimal risky portfolio. The weights of the risk-free asset and the optimal risky portfolio in the optimal investor portfolio depend on the investor's risk tolerance (eg, their indifference curve).
- The optimal investor portfolio is defined by the point where the investor's indifference curve is tangent to the optimal CAL.

Going forward, we assume that all investors have homogeneous (ie, identical) expectations regarding the risk-return distribution for each asset; therefore, only one optimal *risky* portfolio exists. If the investors had different expectations regarding various assets, there would be different optimal risky portfolios.

Capital Market Theory: The Capital Market Line

✓ **LOS:** Explain the capital allocation line (CAL) and the capital market line (CML).

Passive and Active Portfolios

If markets are informationally efficient, a security's market price is the unbiased estimate of the sum of the discounted values of its expected cash flows. In that situation, investors cannot earn a rate of return that exceeds the required rate of return from their investment. Thus, they should adopt passive investment strategies, which typically have lower costs and are easy to implement.

However, investors with more confidence in their ability to forecast cash flows and estimate growth rates and discount rates might consider an active investment strategy. Under this strategy, an investor would make forecasts to determine whether an asset is fairly priced by the market, and then trade on any perceived mispricing.

What Is "The Market"?

For the purposes of this learning module, the term "market" refers to all assets (eg, stocks, bonds, real estate, commodities, etc.) that are tradable and investable; going forward (unless otherwise noted), we will define the term very narrowly as the S&P 500 Index. "Market return" refers to the return on the S&P 500, and "market risk premium" means the US equity risk premium (eg, the difference between the S&P 500's return and the US long-term interest rate).

The Capital Market Line (CML)

The capital allocation line (CAL) includes all combinations of the risk-free asset and *any* risky asset portfolio. In **the capital market line (CML)**, a special case of the CAL, the *market* portfolio is the risky asset portfolio combined with the risk-free asset.

Graphically, the market portfolio occurs at the point where a line from the risk-free asset is tangent to the Markowitz efficient frontier. The market portfolio is the optimal risky asset portfolio, given efficient markets and homogeneous expectations. All portfolios that lie below the CML offer a lower return than portfolios that plot on the CML for each level of risk.

An interesting observation is that the slopes of the CML and CAL are constant even though they represent combinations of two assets. The important thing to note is that they are *not* combinations of two risky assets, but of a risk-free asset and a risky portfolio.

The risk and return characteristics of portfolios lying on the CML can be determined using the risk and return formulas for two-asset portfolios.

To calculate the expected return on portfolios that lie on the CML:

$$E(R_p) = w_f R_f + (1 - w_f)E(R_m)$$

The variance of portfolios lying on the CML is calculated as:

$$\sigma^2 = w_f^2 \sigma_f^2 + (1 - w_f)^2 \sigma_m^2 + 2w_f(1 - w_f)\text{Cov}(R_f R_m)$$

Equation of the CML

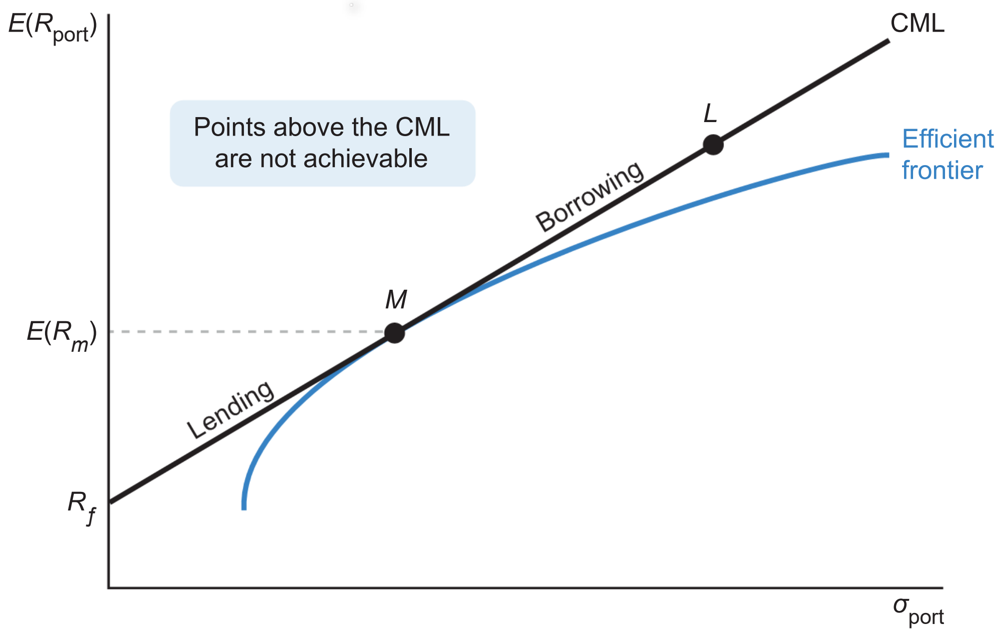
$$E(R_p) = R_f + \frac{E(R_m) - R_f}{\sigma_m} \times \sigma_p$$

In the CML equation, R_f = risk-free rate = the *y-intercept*, and the *slope* of the line—referred to as the *market price of risk*—is:

$$\frac{E(R_m) - R_f}{\sigma_m}$$

In Exhibit 1, the expression relating the expected return of portfolios lying the CML to the portfolios' variance (risk) is similar in derivation to the capital allocation line (CAL) equation The only difference is that the risky asset in the CAL, i, is replaced in the CML by the market portfolio, *m*.

Exhibit 1 Capital market line (CML)



- At point R_f, the investor has invested all funds in the risk-free asset.
- At point *M,* the investor has invested all funds in the market portfolio, which contains only risky securities.
- At any point between R_f and *M*, the investor holds both the market portfolio and the risk-free asset (ie, some of the invested funds are lent at the risk-free rate).

Capital Market Theory: CML – Leveraged Portfolios

LOS: Explain the capital allocation line (CAL) and the capital market line (CML).

An investor may want to attain a higher expected return than available at point *M* (in Exhibit 1), where all funds are invested in the market portfolio. Adding leverage to the portfolio by borrowing money at the risk-free rate and investing it in the market portfolio will allow a risk-return profile beyond (ie, to the right of or above) point *M* on the CML (eg, at point *L*).

The particular point that an investor chooses on the CML depends on the utility function, which is determined by the investor's risk and return preferences.

Example 1 Risk and return of a leveraged portfolio

Sasha Miles, an investor, is evaluating how to allocate funds between the risk-free asset and the market portfolio. She gathers the following information:

- Risk-free rate of return = 6.00%
- Expected return on the market portfolio = 14.00%
- Standard deviation of returns of the market portfolio = 23.00%

Calculate the expected risk and return of a portfolio that is:

1. 75% invested in the market portfolio.
2. 140% invested in the market portfolio.

Solution

Portfolio return and standard deviation can be calculated using the following equations:

$$E(R_p) = w_1 R_f + (1 - w_1)E(R_m)$$

$$\sigma_p = (1 - w_1)\sigma_m$$

1. $E(R_p) = (0.25 \times 0.06) + (0.75 \times 0.14) = 12.00\%$

$$\sigma_p = (0.75 \times 0.23) = 17.25\%$$

2. $E(R_p) = (-0.40 \times 0.06) + (1.40 \times 0.14) = 17.20\%$

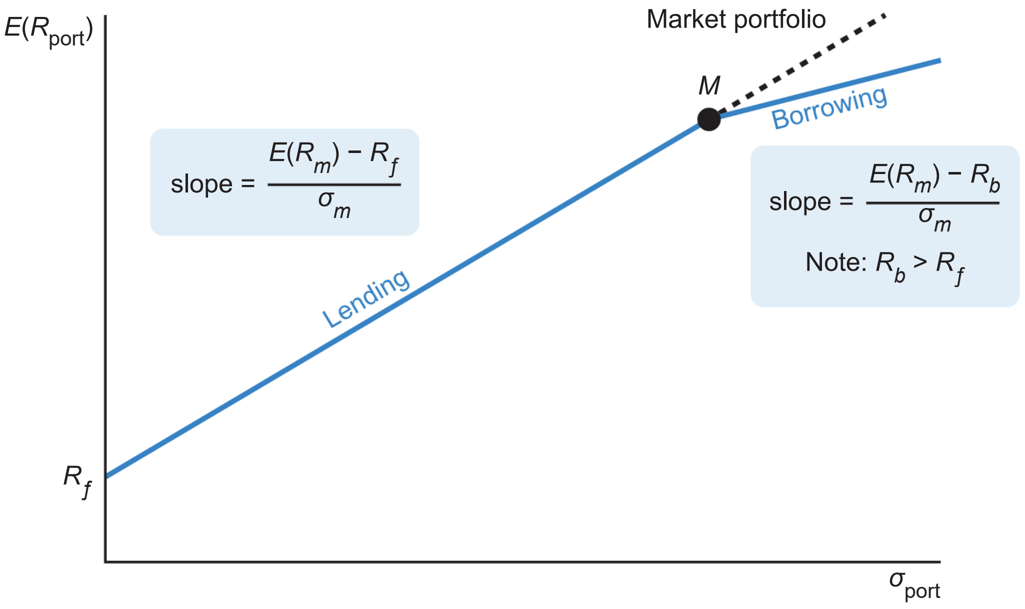
$$\sigma_p = (1.40 \times 0.23) = 32.20\%$$

Recall that the standard deviation of the risk-free asset, as well as the covariance of returns between the risk-free asset and the market portfolio, both equal 0.

A weight of 140% in the market portfolio implies that the investor borrows 40% of the funds at the risk-free rate (6.00%).

In Example 1, we assumed that the investor could borrow or lend unlimited amounts of funds at the risk-free rate. In reality, the investor's ability to repay would be less certain than that of the US government, so the rate at which she could borrow would be higher than the rate at which she could lend. Given the disparity in borrowing and lending rates, the CML would no longer be a straight line, as seen in Exhibit 2.

Exhibit 2 CML with different lending and borrowing rates



The slope of the CML to the left of point *M*, when Miles is investing a portion of her portfolio in the risk-free asset at R_f, would be calculated as:

$$\frac{E(R_m) - R_f}{\sigma_m}$$

and the slope of the CML to the right of point *M*, when Miles is borrowing at R_b, would be:

$$\frac{E(R_m) - R_b}{\sigma_m}$$

where R_f is the lending rate, R_b is the borrowing rate, and $R_b > R_f$.

All passively managed portfolios would lie on the kinked portion of the CML, even though an investor's investment in the risk-free asset may be:

- positive (ie, investor's optimal portfolio lies between R_f and point *M*),
- zero (ie, investor's optimal portfolio lies at point *M*), or
- negative (ie, investor's optimal portfolio lies to the right of point *M*).

The risk and return for a leveraged portfolio is higher than that of an unleveraged portfolio. Further, given that the investor's borrowing rate is higher than the risk-free rate, for each additional unit of risk taken beyond point *M* (where the portfolio is leveraged), the investor gets a smaller increase in expected return than to the left of point *M* (where the portfolio is not leveraged).

Systematic and Nonsystematic Risk

✓ **LOS:** Explain systematic and nonsystematic risk, including why an investor should not expect to receive additional return for bearing nonsystematic risk.

When investors diversify across assets that are not perfectly positively correlated, the portfolio's risk is lower than the weighted average of the individual assets' risks. In the market portfolio, all the risk unique to the portfolio's individual assets has been diversified away.

Nonsystematic risk (also known as unique, diversifiable, or firm-specific risk) is the risk that can be "diversified away" by the portfolio construction process.

Systematic risk (also known as nondiversifiable or market risk) is the risk inherent in the market and/or caused by macroeconomic factors. Systematic risk cannot be eliminated by diversification.

$$\text{Total risk} = \text{Systematic risk} + \text{Nonsystematic risk}$$

Important Points Concerning Systematic and Nonsystematic Risk

- Complete diversification of a portfolio requires eliminating all nonsystematic risk.
- Once nonsystematic risk has been entirely eliminated, a completely diversified portfolio will correlate perfectly with the market.
- The market portfolio does not need to include all risky assets to diversify away nonsystematic risk. Studies have shown that a portfolio consisting of 12 to 30 different stocks can diversify away 90% of nonsystematic risk.