Quantitative Methods

Rates and Returns

Intro and Exam Focus

- Calculating returns
  - Holding period return
  - Averaging: arithmetic vs. geometric vs. harmonic means
  - Time-weighted returns vs. money-weighted returns
  - Non-annual compounding
    - Continuously compounded returns
- Return measures: gross, net, after tax, real, leveraged

Components of Interest Rates

Required nominal interest rate on a security =

real risk-free rate
+ inflation premium } **nominal risk-free rate***
+ default risk premium
+ liquidity risk premium
+ maturity risk premium

*Exact relation: (1 + nominal risk-free rate) = (1 + real risk-free rate)(1 + inflation premium)

Holding Period Return

Holding period return (HPR) is calculated as:

$$\frac{\text{End-of-period value} + CF_1}{\text{Beginning-of-period value}} - 1$$

Example:

A share is purchased for $10 per share and is sold three months later for $11, having collected a dividend of $0.25 per share. What is the HPR for the three-month period?

Solution: 3-month HPR =

-1

Averaging Multi-Period Returns

Arithmetic mean: $\frac{R_1 + R_2 + R_3 + ... + R_n}{n}$

- Arithmetic mean does *not* reflect multi-period compounding
  → Most appropriate for single-period returns

Geometric mean:

$$\sqrt[n]{(1+R_1)(1+R_2)(1+R_3)...(1+R_n)} - 1$$

- Geometric mean *does* reflect multi-period compounding
  → Most appropriate for average return over multiple periods

Averaging Multi-Period Returns: Example

An investment has returns of 15%, 35%, and –25% over three periods. Calculate the arithmetic and geometric mean annual return of the investment.

Solution:

Arithmetic mean return =

Geometric mean return =

Calculator keystrokes:

-3

Harmonic Mean

Used for averaging *ratios*

$$\frac{n}{\frac{1}{X_1} + \frac{1}{X_2} + \frac{1}{X_3} + \ldots + \frac{1}{X_n}}$$

- Harmonic mean gives lower weight to extreme observations ("outliers")
  - →Use harmonic mean when concerned about the impact of outliers

Harmonic Mean: Example

An investor buys $3,000 of a stock at the end of Month 1 at $20 per share, and $3,000 at the end of Month 2 at $25 per share.

What is the average cost per share of stock?

Solution:

= per share

-1

Adjusting for Outliers

Trimmed mean: <u>exclude</u> a stated percentage of most extreme observations

A 1% trimmed mean excludes the top 0.5% and the bottom 0.5% of observations.

Winsorized mean: <u>substitute values</u> for the most extreme observations

A 90% winsorized mean substitutes the 95th percentile for all larger observations and the 5th percentile for all smaller observations.

Comparing Averages

- Harmonic mean ≤ geometric mean ≤ arithmetic mean
  - Difference in means is caused by *volatility* of data
  - Means are equal when all values are the same

Time-Weighted Returns

1 + HPR

$$TWR = \left[\left(\frac{\text{end value}_1}{\text{begin value}_1}\right)\left(\frac{\text{end value}_2}{\text{begin value}_2}\right)\cdots\left(\frac{\text{end value}_N}{\text{begin value}_N}\right)\right]^{\frac{1}{\#YEARS}} - 1$$

Holding periods can be <u>any length</u>

Calculate HPRs for periods between significant external cash flows

Adjust starting/ending values for each period for external cash flows

Number of years not necessarily number of holding periods

Money-Weighted Returns

Money-weighted returns are an IRR measure:

$$CF_0 + \frac{CF_1}{1+MWR} + \ldots + \frac{CF_N}{(1+MWR)^N} = 0$$

Periods must be equal length; use shortest period with no significant cash flows

TWR vs. MWR: CFAI Example

Time	Outflows
0	$200 to purchase first share
1	$225 to purchase second share
Time	**Inflows**
1	$5 dividend received from first share (not reinvested)
2	$10 dividend ($5 per share × 2 shares)
2	$470 received from selling two shares at $235 per share

TWR vs. MWR: Solution

TWR:

Period 1 HRP:

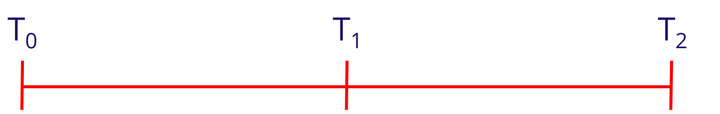
Period 2 HPR:

TWR =

-3

TWR vs. MWR: Solution (cont.)

MWR:



$	$	$
Outflow	Outflow	Outflow
Inflow	Inflow	Inflow
Net	Net	Net

MWR places more weight on second-period returns

0 =

IRR =

-5

Non-Annual Compounding

Convention is to state annualized rates *without* reflecting compounding:

$$R_s = \text{periodic rate} \times m$$

where:

R_s = annualized **stated/quoted** interest rate

m = number of periods per year

Example: If an interest rate is stated as 8% per year compounded semiannually, calculate the effective annual rate (EAR).

Non-Annual Compounding: Solution

If an interest rate is stated as 8% per year compounded semiannually, calculate the effective annual rate (EAR).

8% = semiannual rate × 2

Semiannual rate = 8% / 2 = 4%

$$\text{EAR} = (1 + \text{periodic rate})^m$$

EAR = =

-1

Non-Annual Compounding

Increasing periodicity (R_s = 8%):

Periodicity (m)	EAR
1	8.00%
2	8.16%
4	
12	
365	

-3

Using Non-Annually Compounded Rates: Example

A zero-coupon bond pays €100 in 2 years. What is the price of this bond if interest rates are 6% quoted on a semiannual compounding basis?

Solution:

6% quoted on a semiannual compounding basis implies 6% / 2 = 3% every six months.

→ Price of bond = =

-1

Annualizing Returns

Given an HPR for any holding period:

$$\text{annualized return} = (1 + \text{HPR})^{(\text{year/holding period})} - 1$$

Annualizing Returns: CFAI Example

An investor is evaluating the returns of three recently formed exchange-traded funds. Selected return information on the exchange-traded funds (ETFs) is presented below:

ETF	Time Since Inception	Return Since Inception (%)
1	146 days	4.61
2	5 weeks	1.10
3	15 months	14.35

Which ETF has the highest annualized rate of return?

A. ETF 1.

B. ETF 2.

C. ETF 3.

-4

Continuously Compounded Rates

When compounding frequency *m* gets very large, for a stated rate R_{cc}:

$$HPR = e^{R_{cc}} - 1$$

where:

e = exponential constant 2.71818

Continuously Compounded Rates: Example

If annual continuously compounded quoted rates are 8%, what is the annual HPR?

Solution:

HPR =

Calculator keystrokes:

-1

Continuously Compounded Rates

Solving for quoted continuously compounded rates given actual holding period returns:

$$\ln(1 + HPR) = R_{cc}$$

Continuously Compounded Rates: Example

If an investment grows from $50 to $55, what is the quoted continuously compounded rate of return for the period?

Solution:

1 + HPR =

R_{cc} =

Note that R_{cc} can also be expressed as $\ln(P_T/P_0)$

-2

Using Continuously Compounded Rates

- Continuously compounded rates are *additive* over multiple periods.
- To compound/discount at continuously compounded rates, multiply/divide by $e^{R_{cc}}$.

Example: An investment with an initial value of $1,000 has continuously compounded returns of 5% and –3% in subsequent years. What is the value of the investment after two years?

Solution:

Two-year continuously compounded return =

Future value of the investment after two years =

Calculator keystrokes:

-2

Return Measures

Gross return: return before management fees (after trading expenses)

Net return: return after management fees and administrative expenses

After-tax return: after deducting tax

Real return: after adjusting for inflation

Leveraged return: percentage return on investor's funds

Computation of Returns: CFAI Example

Mr. Lohrmann is analyzing the returns of the Rhein Valley Superior Fund.

In the first year, the fund has a starting value of EUR30 million and reports a net return of 15% for the year.

In the fifth year, the fund reports a net investment return of 3%.

The fund spends a fixed amount of EUR500,000 every year on expenses that are unrelated to the manager's performance.

Mr. Lohrmann has become concerned that both taxes and inflation may reduce his return. Based on the current tax code, he expects to pay 20 percent tax on the return he earns from his investment. Historically, inflation has been around 2 percent and he expects the same rate of inflation to be maintained.

Computation of Returns: Solution

Estimate the annual **gross** return for the first year by adding back the fixed expenses.

Solution:

Net value of fund at year-end: €30,000,000 =

Gross value = + =

Gross return = =

The gross return for the first year is higher by 1.67 percent (€500,000 / €30,000,000) than the 15% investor return reported by the fund. (=).

-4

Computation of Returns: Example (cont.)

What is the after-tax net return for the first year? Assume that all gains are realized at the end of the year and the taxes are paid immediately at that time.

Solution: The net return earned by investors during the first year was 15 percent. Applying a 20 percent tax rate, the after-tax return that accrues to investors is:

=

-1

Computation of Returns: Example (cont.)

What is the after-tax **real** return that investors would have earned in the fifth year?

Solution: The after-tax return earned by investors in the fifth year is:

3% (1 – 0.2) = 2.4%

Inflation reduces the return by 2%, so the after-tax real return earned by i t

=

Note that taxes are paid *before* adjusting for inflation.

-2

Leveraged Returns

$$R_L = R_P + \left(\frac{V_B}{V_E}\right)(R_P - r_D)$$

where:

R_L = return on a leveraged portfolio

R_p = return on assets in portfolio (unlevered)

V_B = value of borrowed funds

V_E = value of equity (investor's funds)

r_D = cost of borrowed funds

Leveraged Returns: Example

An equity portfolio with value of $5 million earns a return of 8%. If the portfolio uses $3 million of borrowed funds at a cost of 4%, what is the leveraged return of the portfolio?

Solution:

$R_L =$ $\qquad\qquad\qquad\qquad =$

-2

Solutions

Holding Period Return

Holding period return (HPR) is calculated as:

$$\frac{\text{end-of-period value} + CF_1}{\text{beginning-of-period value}} - 1$$

Example:

A share is purchased for $10 per share and is sold three months later for $11, having collected a dividend of $0.25 per share. What is the HPR for the three-month period?

Solution: 3-month HPR = [($11 + $0.25) / $10] – 1 = 0.125 or 12.5%

-1

Averaging Multi-Period Returns: Example

An investment has returns of 15%, 35%, and –25% over three periods. Calculate the arithmetic and geometric mean annual return of the investment.

Solution:

Arithmetic mean return = (15% + 35% – 25%) / 3 = 8.33%

Geometric mean return = $\sqrt[3]{((1.15)(1.35)(0.75))} - 1 = 0.052 \text{ or } 5.2\%$

Calculator keystrokes: 1.15 × 1.35 × 0.75 [=] [y^x] 3 [1/x] [=] – 1 [=] 0.052 or 5.2%

-3

Harmonic Mean: Example

Investor buys $3,000 of a stock at the end of Month 1 at $20 a share, and $3,000 at the end of Month 2 at $25 per share.

What is the average cost per share of stock?

Solution:

$$\frac{2}{\left(\frac{1}{20} + \frac{1}{25}\right)} = \$22.22 \text{ per share}$$

-1

TWR vs. MWR: Solution

TWR:

Period 1 HPR: $\left(\frac{\$225 + \$5}{\$200}\right) - 1 = 0.15$ or 15%

Period 2 HPR: $\left(\frac{\$470+\$10}{\$450}\right) - 1 = 0.0667$ or 6.67%

TWR = $\left[(1.15)(1.0667)\right]^{1/2} - 1 = 0.1076$ or 10.76%

-3

TWR vs. MWR: Solution (cont.)

MWR: T_0 — T_1 — T_2

	T_0 $		T_1 $		T_2 $
Outflow	–200	Outflow	–225	Outflow	0
Inflow	0	Inflow	5	Inflow	480
Net	–200	Net	–220	Net	480

$$0 = -200 + \frac{-200}{(1+IRR)} + \frac{480}{(1+IRR)^2}$$

IRR = 9.39%

MWR places more weight on second-period returns

-5

Non-Annual Compounding: Solution

If an interest rate is stated as 8% per year compounded semiannually, calculate the effective annual rate (EAR).

8% = semiannual rate × 2

Semiannual rate = 8% / 2 = 4%

$EAR=(1+\text{periodic rate})^m$

EAR = $(1.04)^2 - 1$ = 0.0816 or 8.16%

-1

Non-Annual Compounding (cont.)

Increasing periodicity (R_s = 8%):

Periodicity (m)	EAR
1	8.00%
2	8.16%
4	8.24%
12	8.30%
365	8.33%

-3

Using Non-Annually Compounded Rates: Example

A zero-coupon bond pays €100 in 2 years. What is the price of this bond if interest rates are 6% quoted on a semiannual compounding basis?

Solution:

6% quoted on a semiannual compounding basis implies 6% / 2 = 3% every six months.

→ Price of bond = €100 / $(1.03)^4$ – 1 = €88.85

-1

Annualizing Returns: CFAI Example

An investor is evaluating the returns of three recently formed exchange-traded funds. Selected return information on the exchange-traded funds (ETFs) is presented below:

ETF	Time Since Inception	Return Since Inception (%)
1	146 days	4.61
2	5 weeks	1.10
3	15 months	14.35

Which ETF has the highest annualized rate of return?

A. ETF 1. Annualized Return = $(1.0461)^{(365/146)} - 1 = 11.93\%$

B. ETF 2. Annualized Return = $(1.011)^{(52/5)} - 1 = 12.05\%$

C. ETF 3. Annualized Return = $(1.1435)^{(12/15)} - 1 = 11.32\%$

-4

Continuously Compounded Rates: Example

If annual continuously compounded quoted rates are 8%, what is the annual HPR?

Solution:

$$HPR = e^{0.08} - 1 = 0.0833 \text{ or } 8.33\%$$

Calculator keystrokes: 0.08 2ND LN – 1 =

-1

Continuously Compounded Rates: Example (cont.)

If an investment grows from \$50 to \$55, what is the quoted continuously compounded rate of return for the period?

Solution:

1 + HPR = \$55/\$50 = 1.1

R_{cc} = Ln(1.1) = 0.0953 or 9.53%

Note that R_{cc} can also be expressed as $ln(P_T/P_0)$

-2

Using Continuously Compounded Rates

- Continuously compounded rates are *additive* over multiple periods.
- To compound/discount at continuously compounded rates, multiply/divide by $e^{R_{cc}}$.

Example: An investment with an initial value of $1,000 has continuously compounded returns of 5% and –3% in subsequent years. What is the value of the investment after two years?

Solution:

Two-year continuously compounded return = 5% – 3% = 2%

Future value of the investment after two years = $\$1{,}000e^{0.02} = \$1{,}020.20$

Calculator keystrokes: 0.02 2ND LN × 1,000 =

-2

Computation of Returns: Solution

Estimate the annual **gross** return for the first year by adding back the fixed expenses.

Solution:

Net value of fund at year end: €30,000,000 × 1.15 = €34,500,000

Gross value = €34,500,000 + €500,000 = €35,000,000

Gross return = (€35,000,000 / €30,000,000) – 1= 16.67%

The gross return for the first year is higher by 1.67 percent (€500,000 / €30,000,000) than the 15% investor return reported by the fund. 16.67% (= 15% + 1.67%).

-4

Computation of Returns: Example (cont.)

What is the after-tax net return for the first year? Assume that all gains are realized at the end of the year and the taxes are paid immediately at that time.

Solution: The net return earned by investors during the first year was 15 percent. Applying a 20 percent tax rate, the after-tax return that accrues to investors is:

$$15\% (1 - 0.2) = 12\%$$

-1

Computation of Returns: Example (cont.)

What is the after-tax **real** return that investors would have earned in the fifth year?

Solution: The after-tax return earned by investors in the fifth year is:

$$3\% (1 - 0.2) = 2.4\%$$

Inflation reduces the return by 2 percent, so the after-tax real return earned by investors in the fifth year is:

$$\frac{(1.024)}{(1.02)} - 1 = 0.0039 \text{ or } 0.39\%$$

Note that taxes are paid *before* adjusting for inflation.

-2