UpperMark™
Study Handbook

CAIA® Level II

Volume 2

Topic 5: Methods and Models

Topic 6: Accessing Alternative Investments

Topic 7: Due Diligence and Selecting Managers

UpperMark™

2026.1025

It is recommended that candidates use any exam preparation product together with the original CAIA curriculum readings.[1]

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Preface

Volume 2 of the UpperMark™ *Study Handbooks* provides a comprehensive and concise account of each learning objective (L.O.) in Topics 5-7 of the CAIA Level II curriculum. The *Study Handbook* is compiled using the reference materials recommended by the CAIA Association and, as in Volume 1, is organized as follows.

- Each Chapter in the curriculum is presented as a separate chapter, keywords are indicated in ***bold italics***, and learning objective sub-bullets are indicated by underlined, capitalized subheadings (e.g., ROLE OF INVESTMENT OBJECTIVES AND CONSTRAINTS).
- The list of keywords and learning objectives is provided at the start of each chapter.
- Space is provided at the end of each chapter for you to record your *Personal Study Notes*.
- A set of practice exam questions is provided at the end of each chapter. A considerably larger set of practice questions (multiple-choice and constructed-response questions, and constructed-response question sets ["Essays"]) is in our *TestBank* software.

Supplementary information is included in footnotes.

The CAIA Equation Exception List is provided in Appendix 2. The first occurrence in this Handbook of each equation on the list is followed by a (***G***) to indicate that it will be *given* on the exam.

We wish you the best with your exam preparation.

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Topic 5

Methods and Models

Topic 5 is composed of seven Readings on different types of financial models, which are essential tools for portfolio management.

1. Reading 5.1 broadly overviews a number of models, including interest rate models and credit risk models.
2. Reading 5.2 describes the binomial tree method for valuation and hedging.
3. Reading 5.3 reviews multi-factor equity pricing models.
4. Reading 5.4 discusses directional investment strategies in which investors alter risk exposures to time market moves based on technical analysis.
5. Reading 5.5 describes multivariate empirical methods (e.g., multiple regression).
6. Reading 5.6 discusses relative value methods with a focus on fundamental analysis.
7. Reading 5.7 discusses, in the context of real estate, the tax consequences of depreciation and transaction- and appraisal-based indices.

Reading 5.1

Modeling Overview and Fixed-Income Models

Financial models express relations that underlie financial instruments and markets, and describe concepts underlying investment and risk management strategies. This Reading describes different types of models, with a focus on interest rate and credit risk models. The interest rate models described are equilibrium, arbitrage-free, and binomial models. The credit risk models described are the structural and reduced-form models, which use different economic approaches to generate the probability of default and credit spread; and a third model that uses a borrower's financial data to generate a credit score for the borrower.

Learning Objectives

5.1.1 Demonstrate knowledge of underlying models of investment strategy.

i. Compare normative models with positive models
ii. Distinguish between theoretical and empirical models
iii. Distinguish between applied versus abstract models
iv. Compare cross-sectional versus time-series models
v. Discuss the importance of methodology in model building

Keywords

1. Abstract models
2. Applied models
3. Cross-sectional models
4. Empirical models
5. Endogenous variable
6. Exogenous variable
7. Normative model
8. Panel data sets
9. Positive model
10. Theoretical models
11. Time-series models

5.1.2 Demonstrate knowledge of equilibrium models and arbitrage-free models of the term structure.

i. Contrast equilibrium fixed-income models with arbitrage-free models
ii. Apply Vasicek's model
iii. Contrast the Cox, Ingersoll, and Ross (CIR) model with Vasicek's model
iv. Discuss the Ho and Lee model

Keywords

1. Arbitrage-free models of the term structure
2. Cox, Ingersoll, and Ross model
3. Equilibrium models of the term structure
4. Ho and Lee model
5. Vasicek's model

5.1.3 Demonstrate knowledge of the Black-Derman-Toy (BDT) model.

i. Interpret a binomial BDT tree
ii. Understand how to calibrate the level of rates based on average returns
iii. Understand how to calibrate the spread of rates based on volatilities
iv. Discuss BDT calibrations in general

Keyword

1. Black-Derman-Toy Model (BDT model)

5.1.4 Demonstrate knowledge of credit risk and credit risk modeling.

i. Distinguish types of credit events that may lead to an increase in credit risk
ii. Explain exposure at default (EAD) and loss given default (LGD)
iii. Describe how adverse selection and moral hazard relate to credit risk
iv. Discuss how probability of default (PD) and recovery rate (RR) affect credit risk
v. Calculate loss given default and expected loss from credit risk
vi. Describe the basic concepts of credit risk modeling
vii. Contrast the three approaches to credit risk modeling

Keyword

1. Credit events

5.1.5 Demonstrate knowledge of the structural model approach to credit risk.

i. Describe Merton's structural model using the option-like nature (both call options and put options) of traditional corporate securities
ii. Describe the inherent conflict of interest that exists between shareholders and bondholders
iii. Evaluate advantages and disadvantages of the Merton Model
iv. Understand how binomial trees can be used to value structured products

5.1.6 Demonstrate knowledge of the Merton model.

i. Apply the Merton model to determine equity values and payoffs to bondholders for a given investment
ii. Calculate the value of risky debt using the Black-Scholes option pricing model in the Merton model to price a given firm's equity as a call option on the stock of the underlying company
iii. Evaluate the use of Black-Scholes option pricing in the Merton model
iv. Analyze the role of credit spreads in structural models and how the credit spread can be used to calculate the price of risky debt
v. Understand the four important properties of the Merton model

5.1.7 Demonstrate knowledge of the Kealhover, McQuown, and Vasicek (KMV) credit risk model.

i. Describe the characteristics and application of the KMV model
ii. Evaluate the credit score (the distance to default) for a given firm using the KMV model
iii. Evaluate the expected default frequency for a given investment using the KMV model

Keywords

1. Default trigger
2. Distance to default (DD)
3. Expected default frequency (EDF)
4. KMV model

5.1.8 Demonstrate knowledge of reduced-form models.

i. Describe the characteristics of reduced-form models
ii. Discuss the role of default intensity in reduced-form models and calculate default intensity for a given firm
iii. Demonstrate how default intensity can be incorporated into the valuation of risky debt
iv. Analyze the relationship among credit spreads, default intensities, and recovery rates, and use two of these factors as variables to solve for the third for a given investment
v. Describe the two predominant reduced-form credit models

Keyword

1. Default intensity

5.1.9 Demonstrate knowledge of empirical credit models.

i. Contrast empirical credit models with structural and reduced-form models
ii. Describe the purpose and characteristics of the Altman Z-score model
iii. Understand the five financial ratios that are used as inputs to determine Altman Z-scores
iv. Evaluate Z-scores in Altman's credit scoring model

Keywords

1. Credit score
2. Z-score model

L.O. 5.1.1 **DEMONSTRATE KNOWLEDGE OF UNDERLYING MODELS OF INVESTMENT STRATEGIES.**

Variables used in models may be classified as exogenous or endogenous.

- An ***exogenous variable*** is a value that is determined outside a model and thus taken as given.
- An ***endogenous variable*** is determined inside a model and thus takes on the value the model prescribes.

For instance, in an endowment fund's cash management model, an exogenous variable may be the amount of cash received from donations and income from investments, and endogenous variables may be decision variables such as the amount of money invested in new deals.

By understanding different types of models, asset allocators may form portfolios that are better diversified across types of models. Four common distinctions of models are discussed in this L.O.: 1) normative vs. positive, 2) theoretical vs. empirical, 3) applied vs. abstract, and 4) cross-sectional vs. time-series.

1. NORMATIVE VS. POSITIVE MODELS

   Models may be classified as normative or positive.

   - ***Normative models***
     - Normative economic models aim to describe how market participants and asset prices should behave. For instance, arbitrage-free models describe relationships that should hold given that arbitrageurs' actions will eliminate arbitrage opportunities. For example, a normative strategy is a strategy based on put-call parity.
     - These models are used to identify driving factors of rational financial decisions based on idealized assumptions and conditions.
     - They may also be used to identify potential mispricings by identifying how assets should be priced. Normative reasoning assumes that actual prices converge toward prices predicted by a normative model.
   - ***Positive models***
     - Positive economic models explain/predict how market participants and asset prices actually behave. For instance, technical trading is based on positive models. For example, a positive strategy is a strategy based on point-and-figure charts.[1]
     - These models are often used to identify potential mispricings by identifying patterns in actual price movements.

   The primary criterion for assessing models is their ability to predict the future. The reality of assumptions or ability to explain the past is of secondary importance. Normative and positive models can be used to predict future behavior and for analyzing alternative investments. Arguably, the key aspect of analyzing many trading strategies is knowing whether the underlying model is based on how prices should or do behave. For instance, if a fund manager implements a trade to benefit from a forecasted change in prices, did the manager base the forecast on how prices should behave or by observing how past prices behaved?

[1] Point-and-figure charts describe price movements without regard for the passage of time. The charts consist of columns of X's and O's, where the X columns represent rising prices and the O columns represent falling prices. No change in a point-and-figure chart indicates no movement in price.

2. <u>THEORETICAL VS. EMPIRICAL MODELS</u>

Related to normative and positive models are theoretical and empirical models. The choice between a theoretical and an empirical model generally depends on the complexity of the underlying relationships and the reliability of the data.

- Theoretical models
  - These models describe behavior based on assumptions that reflect well-established underlying behavior.
  - They provide a reasonable explanation of simple behavior, but are not practical for securities with complex attributes and relationships. A single theoretical model does not exist that can explain all relationships in different markets.
  - An application of theoretical models includes theoretically determining the price of an option based on assumptions such as perfect markets, stock prices that follow a particular process, and absence of arbitrage.
- Empirical models
  - These models describe behavior based on observations of historical data. They require underlying variables to be relatively constant or to change in a predictable way. They also require large data sets to produce reasonable results.
  - Empirical models are often used to explain complex behavior. As such, they are most effective for alternative investments due to the investments' illiquidity, time-varying risks, and use of dynamic strategies.
  - Applications of empirical models include analyzing complex securities with option features and approximating the relationship between observed prices of options and their underlying variables.

3. <u>APPLIED VS. ABSTRACT MODELS</u>

Models may be distinguished as applied or abstract.

- ***Applied models***
  - These models are used for solving real-world problems. For instance, the Markowitz model of portfolio management is an applied model that provides a useful approach to achieve diversification efficiently.
  - Most asset pricing models used in traditional and alternative investing are applied.
- ***Abstract models*** (or basic models)
  - These models are typically theoretical and explain hypothetical behavior in unrealistic situations. They do not address real-world problems.
  - For instance, an abstract model might describe how two people trade securities in a world with only two people and two risk factors.

4. <u>CROSS-SECTIONAL VS. TIME-SERIES MODELS</u>

Models may be classified as cross-sectional or time-series models.

- ***Cross-sectional models***
  - These models analyze relationships across variables observed at a single point in time (e.g., using investment returns to explain differences in risk premiums).
- ***Time-series models***
  - These models analyze the behavior of an asset or a set of assets across time.

Models may also be both cross-sectional and time-series. This type of model is referred to as a panel study and using data composed of multiple assets over multiple time periods. The data are referred to as ***panel data sets***, cross-sectional time-series data sets, or longitudinal data.

For instance, time-series and cross-sectional models may be used to analyze returns on real estate investment trusts (REITs) and on a REIT index.

- A time-series model may be constructed that explains the REIT index returns over time by regressing the index returns against mortgage rates and stock returns. A cross-sectional model is then used to explain why various REITs have different returns by regressing individual REIT returns against variables such as region and property type.
- Alternatively, all returns for each time period and for each REIT can be analyzed using a single regression model in a panel study.

Some asset pricing models may be classified in more than one way. For instance, abstract models tend to be normative and theoretical, while applied models tend to be empirical and positive. In some studies, complementary modeling approaches may be combined. For instance, a theoretical model can be designed and then tested in an empirical framework. Other examples of models with multiple distinctions are provided below.

- Theoretical, normative, and applied model – An analyst identifies a profitable trading opportunity by specifying an asset's equilibrium price and recommending trades when the asset's actual price deviates from its equilibrium price.
- Empirical, positive, and applied model – An analyst identifies a statistical trading pattern and uses the pattern to generate trading signals.

L.O. 5.1.2 DEMONSTRATE KNOWLEDGE OF EQUILIBRIUM MODELS AND ARBITRAGE-FREE MODELS OF THE TERM STRUCTURE

Models of the term structure of interest rates are used to describe the evolution of default-free bond values and to value fixed-income derivatives. There are two broad approaches to modeling the term structure of interest rates: equilibrium models and arbitrage-free models. This L.O. focuses on these models.

Equilibrium models of the term structure (also referred to as first-generation models) make assumptions about the structure of fixed-income markets and then model bond prices and the term structure of interest rates based on economic reasoning. The equilibrium models discussed in this L.O. model the entire term structure of interest rates (i.e., the yield on long-term bonds) by taking the short-term interest rate process as given (i.e., assuming a process for the short-term rate) and assuming that the unbiased expectation hypothesis holds for bond prices (which implies that credit risk-free bonds of all maturities have the same expected return over the short term).

The first section of this L.O. discusses two equilibrium models: Vasicek (1977) and Cox, Ingersoll, and Ross (1985).

VASICEK MODEL

Vasicek's model is a single-factor model of the term structure that assumes constant volatility and that the short-term interest rate drifts toward a specific long-term mean. The model describes the mean-reverting process for the short-term interest rate as:

$$\tilde{r}_{t+1} = r_t + \kappa(\mu - r_t) + \sigma\tilde{\varepsilon}_{t+1},$$

where $\tilde{r}_{t+1}$ is the next period's short-term rate, r_t is the current short-term rate, μ is a positive constant that represents the short-term rate's long-term mean value, κ is a positive constant that determines the speed of adjustment to the long-term mean, σ is the volatility of changes in interest rates, and $\kappa(\mu - r_t)$ and $\sigma\tilde{\varepsilon}_{t+1}$ are two adjustments.

- The first adjustment $\kappa(\mu - r_t)$ indicates that the next period's short-term rate will be higher than the current rate if $\mu > r_t$.
  - Since the short-term rate is mean-reverting (i.e., reverting to its long-term mean), the short-term rate is likely to increase if it is currently below its long-term mean and likely to decrease if it is above its long-term mean.
  - The larger the κ, the faster the short-term rate approaches its long-term mean.
- The second adjustment factor $\sigma\tilde{\varepsilon}_{t+1}$ represents the unexpected change in the short-term rate, where $\tilde{\varepsilon}_{t+1}$ represents a noise factor that is assumed to be a standardized normally distributed random variable (i.e., with a mean of zero and a standard deviation of one).

  The Vasicek model may be expressed in terms of the next period's expected short-term rate as:

$$E[r_{t+1}] = r_t + \kappa(\mu - r_t).$$

This shows that the expected change in the short-term rate is given by $\kappa(\mu - r_t)$.

A criticism of the Vasicek model is that it assumes the volatility of changes in interest rates is constant as the level of interest rates changes (e.g., volatility of interest rate changes is always 1.2%). A consequence of this that the model may produce negative interest rates, which is another criticism of the model.

Example

In the Vasicek model, the long-term mean of the short-term interest rate is 6% and the speed of adjustment parameter is 0.75. If the current short-term rate is 4.3%, what is the expected short-term rate in the next period?

Expected short-term rate in next period is:

$$E[r_{t+1}] = r_t + \kappa(\mu - r_t) = 4.3\% + 0.75(6\% - 4.3\%) = 5.575\%.$$

Vasicek model and the term structure of interest rates

In the Vasicek model, all bond prices are driven by the short-term interest rate, which implies that the only source of uncertainty in the bond market is the random change in the short-term rate. Formulas for bond prices may be determined in different ways. For instance, assuming that the unbiased expectations hypothesis holds for the term structure, all credit-risk-free bonds are expected to earn the same rate of return in the short-term. Based on this assumption, a formula can be derived for the yield to maturity of a zero-coupon bond and for the term structure of interest rates (or yield curve) in the Vasicek model.

The Vasicek model can generate term structures of different shapes: downward-sloping, upward-sloping, and humped. The slope of the curve is driven by the relationship between the short-term rate and the long-term mean rate. The hump reflects investors' risk aversion.

COX, INGERSOLL, AND ROSS MODEL

The ***Cox, Ingersoll, and Ross*** (CIR) ***model*** modifies the Vasicek model so that the variance of the short-term rate is proportional to the short-term rate.[2] As a result, the CIR model does not allow negative interest rates, since, as rates approach zero, their volatility approaches zero. The CIR model is a single-factor model that describes the short-term interest rate process as:

$$r_{t+1} = r_t + \kappa(\mu - r_t) + \sqrt{r_t}\sigma\tilde{\varepsilon}_{t+1}, \qquad (G)^3$$

where the three constant parameters κ, μ, and σ are as defined in the Vasicek model.

- The relationship that the variance of the short-term rate is proportional to the rate itself is observed empirically: volatility of interest rate changes is higher when the short-term rate is relatively high.
- Like the Vasicek model, the CIR model can generate yield curves of different shapes.

ARBITRAGE-FREE MODELS OF THE TERM STRUCTURE

Arbitrage-free models of the term structure (also referred to as second-generation models) generate bond prices that do not allow for arbitrage opportunities. Under risk-neutral modelling, returns on all investments should equal the short-term rate. Arbitrage-free models also use the currently observed term structure to determine the parameters of the model, which results in a theoretical term structure model that is consistent with the observed term structure. As a result, any fixed-income derivative instrument priced using this model will be consistent with the current term structure and will preclude arbitrage opportunities involving the derivatives and available bonds.

This section describes the Ho and Lee (1986) arbitrage-free model, which was the first arbitrage-free model of interest rates developed.

[2] In other words, volatility equals σ in the Vasicek model and equals $\sqrt{r_t}\sigma$ in the CIR model.

[3] As stated in the Preface, the CAIA Association may provide specific equations on the exam. Equations that may be provided are indicated in our Handbooks with a (G) and are in Appendix 2 of this *Handbook*.

HO AND LEE MODEL

The ***Ho and Lee model*** is a single-factor model that assumes that the short-term interest rate follows a normally distributed process, with a drift parameter selected so that the modeled term structure of interest rates fits the observed term structure. The Ho and Lee model describes the short-term rate as:

$$r_{t+1} = r_t + \theta_t + \sigma\tilde{\varepsilon}_{t+1}\,, \qquad (G)$$

where θ_t is a time-dependent mean change in the short-term rate (selected to ensure that the model fits the initial term structure of interest rates), σ is the constant standard deviation of changes in the short-term rate, and $\tilde{\varepsilon}_{t+1}$ is a binomial random variable (which is either +1 or –1). In contrast to Vasicek's model that has constant parameters, the θ_t parameter in the Ho and Lee model is not constant, but is determined by the current term structure of interest rates.

Ho and Lee use this model to determine bond prices using a binomial pricing approach, in which current zero-coupon bond prices are taken as given and used to value the parameters of the model based on the current term structure of interest rates. The term structure is then assumed to be affected by random changes in interest rates. Bond prices evolve in response to random changes in interest rates. The model uses the concept that, with risk-neutral probabilities, the bond price in every state equals the bond's expected value in the next period discounted at the riskless rate to obtain analytical solutions for the bond price in each future state.

Since the Ho and Lee model is calibrated to fit the currently observed term structure of interest rates, resulting prices of interest rate instruments (e.g., callable bonds and bond options) are consistent with the current term structure. This implies that the actions of arbitrageurs can ensure that derivative prices are tied to observable bond prices so that arbitrage profits are not possible.

The key disadvantages of the Ho and Lee model are that it assumes a simple binomial process for bond prices and it can produce negative interest rates. Variations of this model have been developed that prevent negative interest rates and use more sophisticated processes for bond prices.

L.O. 5.1.3 **DEMONSTRATE KNOWLEDGE OF THE BLACK-DERMAN-TOY (BDT) MODEL.**

The ***Black-Derman-Toy (BDT) model*** (Black and Toy 1990) is an interest rate model that constructs no-arbitrage interest rate trees using both the observed term structure of interest rates and rate volatilities (i.e., implied volatilities of interest rate caplets). The model may be used with any compounding assumption to model spot rates, forward rates, and/or discount factors, and used to find no-arbitrage values of fixed-income derivatives. As described below, the BDT model focuses on two relations: average forward rates and interest rate volatilities.

BDT BINOMIAL TREE

A BDT binomial tree represents short-term (1-year) spot rates, where, at each rate in the tree, there are two possible rates next year that each occur with a probability of 0.5 (or 50%). For instance, as depicted, in Figure 1, for a 2-year binomial tree with a current 1-year spot rate of r_0, there are two possible 1-year spot rates next year: r_u in the up node and r_d in the down node.

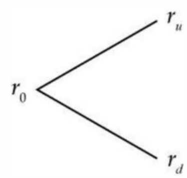

Figure 1: BDT Binomial Tree

The BDT tree is calibrated to match zero-coupon (discount) bond yields and a set of volatilities. Specifically, the two possible future rates (e.g., r_u and r_d) are calibrated based on two key constraints.

1. The average 2-year return of the two paths must equal the return of the 2-year zero-coupon bond.
2. The spread between the up and down rates must be consistent with the implied rate volatility of the short-term rate from a 1-year caplet on the short-term rate.

A discussion of these constraints follows.

Calibration involves adjusting a model's potential outcomes to prevent arbitrage opportunities given observed market rates and prices. For instance, the values r_u and r_d are calibrated (found) as the only no-arbitrage values that are consistent with a 2-year zero-coupon bond yield and the volatility of the short rate. This is illustrated below.

CALIBRATING THE LEVEL OF RATES BASED ON AVERAGE RETURNS

The rates in a BDT tree are calibrated so that the averaged future value of the paths based on rolling over *n* single-period bonds equals the observed total return on an *n*-year zero-coupon bond. For instance, the averaged return of the two paths of investing in the short-rate for two years may be expressed as:

$$\text{Averaged short-rate total return} = 0.5(1+r_0)\left[(1+r_u)+(1+r_d)\right]-1,$$ [4]

where calibration of the rates r_u and r_d is based on the yield of a 2-year zero-coupon bond.

[4] In the CAIA reference text, this formula is given as:

$$\text{Averaged short-rate total return} = 0.5\left[(1+r_0)(1+r_u)+(1+r_0)(1+r_d)\right]-1.$$

We simplify the equation by factoring out $(1+r_0)$.

Example 1

An analyst, calibrating a Black-Derman-Toy binomial tree model, obtains 4% for the current short rate and 3.2% and 6.5% for the two potential rates after one year.

a) What is the total 2-year return obtained from investing in the current 2-year short rate, assuming annual compounding?

b) What is the annual yield on a 2-year zero-coupon bond that generates the total 2-year return?

a) Averaged short-rate total return $= 0.5(1+r_0)\left[(1+r_u)+(1+r_d)\right]-1$

$$= 0.5(1+0.04)\left[(1+0.065)+(1+0.032)\right]-1 = 9.044\%$$

Thus, the total 2-year return from investing in the current 2-year short rate is 9.044%.

Note 1: Since r_u and r_d are calibrated (found) so that the averaged short-rate total return equals the 2-year return on a zero-coupon bond, this means that the 2-year return on the zero-coupon bond is 9.044%.

b) Yield on 2-year zero-coupon bond that generates a total 2-year return of 9.044% is:

Annualized two-year return $= \sqrt{1+0.09044} - 1 = 4.424\%$.

Note 2: If you were given the yield on a 2-year zero-coupon bond as being 4.424%, you could find the 2-year return on the zero-coupon bond as:

2-year return on zero-coupon bond $= (1+0.04424)^2 - 1 = 9.044\%$.

CALIBRATING THE SPREAD OF RATES BASED ON VOLATILITIES

The second constraint in the BDT model is that the spread between the possible values of the short rate next year (i.e., r_u and r_d) must be consistent with the observed volatility of the short-term rate on a one-year caplet. To this end, the relationship between r_u and r_d may be expressed as:

$$r_u = r_d e^{2\sigma},$$

where σ is the implied volatility (as a continuous rate) of the one-year caplet on the short-term rate.

Example 2

An analyst, calibrating a Black-Derman-Toy two-period binomial tree model, observes that the second-period short rate in the down state is 3.8% and the implied continuous volatility of the short rate in the second period is 14%. What is the value for the second-period short rate in the up state?

$$r_u = r_d e^{2\sigma} = 0.038e^{2\times 0.14} = 0.038 \times 1.3231 = 0.0503 = 5.03\%$$

BDT CALIBRATIONS IN GENERAL

As discussed, two conditions identify the rates in the second period of a 2-period BDT binomial tree.

1. Spot rates in the current term structure drive the *levels* of the rates projected in the binomial tree.
2. Implied volatilities of options trading on short-term rates (i.e., interest rate caplets) drive the *spreads* between the up and down rates corresponding to the caplets' expiration dates.

The computation associated with finding these rates may be complex, particularly for many binomial trees that typically have at least 30 time periods.

L.O. 5.1.4 **DEMONSTRATE KNOWLEDGE OF CREDIT RISK AND CREDIT RISK MODELING**

CREDIT EVENTS THAT MAY LEAD TO INCREASED CREDIT RISK

Credit refers to funds given to a borrower/debtor by a lender/creditor and may take the form of bank loans, corporate bonds, or government (sovereign) bonds. Credit risk refers to the risk of loss resulting from a ***credit event*** with a counterparty, which may include the following.

- Bankruptcy – This occurs if an entity becomes insolvent or is dissolved and is thus unable to meet its financial obligations.
- Downgrading of credit rating – This refers to an entity's credit rating being lowered by credit rating agencies due to changes in the entity's financial condition or broad economic conditions.
- Failure to make timely payments – This refers to a borrower that fails to make timely payments (interest or principal).
- Corporate events – This refers to events (e.g., mergers or spin-offs) that weaken an entity's financial condition, making it difficult for the entity to meet its financial obligations.
- Government actions – This refers to government restrictions (e.g., capital controls) that may prevent a borrower from meeting its obligations.

EXPOSURE AT DEFAULT AND LOSS GIVEN DEFAULT

The degree and effect of credit risk depend on several factors, two of which are exposure at default (credit exposure) and loss given default.

1. Exposure at default (EAD) – This refers to a creditor's potential loss resulting from a credit event.
2. Loss given default (LGD) – This takes into account potential recovery in the event of default.

If 100% of EAD is lost in the event of default, then LGD = EAD. Most often, the creditor recovers some of the losses, so LGD is typically less than EAD.

ADVERSE SELECTION & MORAL HAZARD, & THEIR RELATIONSHIP WITH CREDIT RISK

Adverse selection refers to an economic process in which undesirable outcomes occur when parties to a transaction have asymmetric information (before a financial transaction is completed).

- In the case of debt, borrowers have more information than lenders about their ability to meet their financial obligations. To compensate for this disadvantage, lenders may raise loan rates. However, this action has unintended consequences: borrowers with poor credit quality may still find the higher borrowing cost attractive; and if the lender were to raise interest rates further, the proportion of borrowers with poor credit quality would rise, thus increasing the lender's credit risk. At the extreme, a high interest rate will drive all borrowers away, leaving the lender with unused capital.
- Akerlof (1970) describes a market with asymmetric quality information as a "market for lemons", using the slang word lemon to refer to a defective or faulty item (e.g., a defective or poor-quality used car). Akerlof explains that the implication of asymmetric information between sellers and potential buyers is that the quality of goods in a market declines: poor-quality cars drive high-quality cars from the market. This is because buyers, concerned that a car is of poor quality, will not a pay high price for it; and sellers will not sell high-quality cars at low prices. Therefore, only poor-quality cars will be sold.[5] Consequently, economists use markets for lemons as an example of potential market failure.
- To reduce the effect of adverse selection and the potential problem of market failure, lenders consider borrowers' credit histories and reputations, may require collateral, and may limit the size of the loans.

Moral hazard occurs after an economic transaction is completed and arises when one party to a transaction changes its behavior (e.g., assumes more risk) and the other party bears the consequences.

- For instance, after completing a loan transaction, a borrower may use the loan proceeds to pay dividends to its shareholders or invest in risky projects, which may weaken the borrower's financial condition and increase the lender's default risk. Also, when financial lending institutions can raise money at a very low interest rate (since their deposits are insured by a government agency) and use the funds raised to make risky loans to earn a large spread, the government and the public bear the costs if the borrowers of the risky loans default.
- To reduce moral hazard, lenders monitor borrowers' behavior, restrict how loans' proceeds may be used, and limit the size of loans to risky borrowers.

EFFECT OF PROBABILITY OF DEFAULT AND RECOVERY RATE ON CREDIT RISK

In addition to EAD and LGD, the probability of default (PD) [i.e., probability that a borrower cannot meet its financial obligations] affects credit risk. A borrower's PD is affected by firm-specific and macroeconomic conditions, and can be reduced by mitigating the effects of adverse selection and moral hazard.

Lenders use market data, credit ratings, and experience to attain accurate estimates of the PD. Some credit risk models improve their estimates of PD using market data such as credit spreads (i.e., high yields on credit risky instruments given to lenders). Some lenders improve their PD estimates using historical credit ratings from external rating agencies. For instance, if, historically, 0.1% of AAA-rated firms have defaulted on their loans and this figure is relatively stable, the current rating of a AAA bond can be used to estimate its PD as 0.1%.

[5] In essence, the uninformed buyer's price creates an adverse selection problem that drives high-quality cars from the market.

CALCULATING LGD AND EXPECTED LOSS FROM CREDIT RISK

The recovery rate (RR) is the percentage of EAD that may be recovered after default. Since it may take years to access the recovered amount, the RR is calculated using the present value of the recovered amount and may be expressed as:

$$RR = \frac{\text{Present value of sum to be recovered}}{\text{EAD}},$$

where EAD may be expressed as the sum of principal and interest due.

LGD may be expressed in terms of RR as:

$$LGD = EAD(1 - RR),$$

or expressed as: LGD = EAD − Present value of sum to be recovered. The expected loss from credit risk may be expressed as:

$$E[\text{Loss}] = LGD \times PD = EAD(1 - RR) \times PD.$$

Example

A loan's exposure at default at the end of the year is $150 million. If the loan defaults, the lender expects to recover $60 million three years after default. The borrower's probability of default is estimated to be 1.8%. If the appropriate annual discount rate is 7%, what is the:

a) Loan's recovery rate?
b) Loss given default?
c) Lender's expected loss?

a) $RR = \frac{\text{PV of sum to be recovered}}{\text{EAD}} = \frac{\text{PV of \$60m}}{\$150\text{m}}$

$$\text{PV of \$60m} = \frac{\$60\text{m}}{(1+0.07)^3} = \$48.98\text{m}$$

$$\Rightarrow RR = \frac{\text{PV of \$60m}}{\$150\text{m}} = \frac{\$48.98\text{m}}{\$150\text{m}} = 32.65\%$$

b) $LGD = EAD(1 - RR) = \$150\text{m}(1 - 0.3265) = \$101.02\text{m}$

c) $E[\text{Loss}] = LGD \times PD = \$101.02\text{m} \times 1.8\% = \$1.8184\text{m}$

CREDIT RISK MODELING

Owners of government bonds (e.g., U.S. Treasuries) are exposed to interest rate risk, but not credit risk, since most governments are unlikely to default on their debt obligations. In contrast, owners of bonds issued by levered entities are exposed to interest rate risk and credit risk.

In the extreme case of default, a company may be unable to continue as a going concern and close its operations. However, in practice, default is not a terminal event: companies do not cease to exist in the event of default. Instead, many companies experiencing distress undergo a restructuring process to change their business and enable them to emerge as going concerns. Companies that cannot restructure may need to liquidate their operations.

APPROACHES TO CREDIT RISK MODELING

Credit risk models can be used to assess credit-based instruments to determine whether their relative prices are correct. Based on this, investors can take positions in the instruments. Most credit risk models assume that default is an end point.

There are three types of credit risk modeling approaches.

1. Structural approach
   - This approach assumes an explicit relationship between a firm's capital structure and default, and describes the value of a firm's assets as being equal to the value of its equity plus the value of its debt.
     - ➢ The firm's equity is considered a call option on its assets, with a strike price equal to the face value of its debt due at exercise date.
     - ➢ The firm's risky debt is considered a risk-free bond and a short position in a put option on the firm's assets. If the assets' value is less than the debt's face value, the put option will be exercised on the bondholders, resulting in their giving up the risk-free bond and receiving the firm's assets.
2. Reduced-form approach
   - This approach models default as an exogenous event driven by a random signal, the behavior of which is the deciding factor of default.
   - These models assume that default is a random event that may be described using statistical and economic models.
3. Empirical approach
   - This approach does not attempt to model companies; instead, it examines the financial data of companies that have defaulted to understand their credit risk. Based on this, the approach generates a credit score that is used to rank firms based on their creditworthiness.

L.O. 5.1.5 **DEMONSTRATE KNOWLEDGE OF THE STRUCTURAL MODEL APPROACH THROUGH THE LENS OF THE MERTON MODEL**

MERTON'S STRUCTURAL MODEL

The structural approach to pricing credit risk was developed by Robert Merton, who recognized that the simplified capital structure of a traditional operating company and associated credit risk have option-like characteristics, which facilitates valuation of credit risk.

The structural model assumes that a levered operating firm has a simple capital structure consisting of two securities: one issue of zero-coupon debt and one class of non-dividend-paying equity. Under the call option view of capital structure, the equity in a leveraged firm is equivalent to a long call option on the firm's assets, with an exercise price equal to the face value of the firm's debt and an expiration date equal to the maturity date of the debt.

Equity of levered firm = Call option on firm's assets

- If the firm performs well, it repays its debt holders at maturity and its shareholders own the firm's assets (i.e., they pay the option's strike price amount owed on the debt to claim the underlying assets).
- If the firm performs poorly, the shareholders can declare bankruptcy (i.e., not exercise their option to take ownership of the firm's assets) and the debt holders receive the firm's assets.

Equity holders, as call option holders, enjoy unlimited upside potential from gains in the firm's asset and have limited loss exposure since they can allow the option to expire unexercised.

The firm's debt can also be described in terms of options.

- If the value of the firm's assets equals the sum of its equity and debt, and equity is a call option on the firms' assets, then the firm's debt is equivalent to owning the firm's assets and writing a call on the assets. Thus, owning debt is the same as owning a covered call (i.e., long assets and short a call on the assets).
- Based on put-call parity, a call option can be considered a long put option with the underlying assets financed with a riskless bond.[6] This leads to the put option view of capital structure, which views the levered firm's equity holders as owning the firm's assets via riskless financing and having a put option to deliver the assets to the firm's debt holders. Thus, the levered firm's risky debt is equivalent to owning a riskless bond and writing a put on the firm's assets (such that the equity holders can put the firm's assets to the debt holders in exchange for the debt).

Debt of levered firm = Riskless bond − Put option on firm's assets

  - Another way to consider the value of the firm's risky debt is as the value of an otherwise identical riskless bond reduced by the value of a put, where the reduction in the risky debt's value represents the market's price for bearing the firm's credit risk.
  - The put option represents the equity owners' ability to put the firm's assets to the debt holders by declaring bankruptcy and benefiting from limited liability.
  - If the firm performs poorly, the debt holders suffer losses since they must pay the stockholders a strike price equal to the value of the riskless bond.
  - If the firm defaults, the debt holders receive only the reduced value of the underlying assets. This risk is captured by the debt holders' short put position.

[6] Put-call parity may be expressed as: Assets = Call + (Riskless bond – Put).

Given that both call and put option views of the levered firm describe the value of a firm's assets in terms of options on the assets, option pricing models such as Black-Scholes can be used to generate prices for the firm's assets. Implementing the structural approach involves using market prices to find values of the model's parameters (e.g., asset volatility) and inserting the parameters into the structural model to generate prices for the credit-risky assets.

INHERENT CONFLICT OF INTEREST BETWEEN BONDHOLDERS & STOCKHOLDERS

There is an inherent conflict of interest between a firm's bondholders and stockholders with respect to the desired riskiness of the firm's assets: bondholders prefer less risk (due to their short put position), whereas stockholders (as long call option holders) prefer more risk since, all else equal, the time value of their call option is higher when asset volatilities are higher.

- The firm's equity value at the debt's maturity date is the greater of zero and the difference between the value of the firm's assets and the face value of debt. Therefore, particularly when the credit risk of the firm's debt is high (when the firm's asset value is near or below the face value of its debt), stockholders have an incentive to prefer risky projects because they benefit more when the risky projects succeed, but it is the bondholders who lose if the projects fail.
- The conflict of interest in a firm's capital structure represents a zero-sum game in which managers can shift wealth from bondholders to stockholders by increasing the riskiness of the firm's assets or shift wealth from stockholders to bondholders by reducing the assets' riskiness. Similarly, in many structured products, this conflict of interest exists as wealth can be transferred from senior to junior tranches by increasing the riskiness of collateral assets.

USING BINOMIAL TREES TO VALUE STRUCTURED PRODUCTS

An alternative to using the Black-Scholes option pricing model to implement the structural model is to use the binomial option pricing model. Binomial tree models, which are based on risk-neutral pricing, are effective tools for valuing securities with embedded options. In fact, the Black-Scholes option pricing model is typically used for valuation of simple options.

For instance, with a binomial model, the value of credit-risky securities of a firm or in a structured product can be estimated using two binomial trees: one for the value of the underlying assets and one for the interest rate. The process involves estimating future cash flows based on the asset values and then pricing the credit-risky securities using backward induction.

EVALUATION OF MERTON MODEL

Merton's model has several intuitive properties and serves as a basis for more complex models. For instance, the risk-neutral probability of default (i.e., the Q-measure) may be expressed as:

$$\Pr\left(A_T \leq K\right) = 1 - N\left(d - \sigma_A \sqrt{\tau}\right), \qquad \textbf{(G)}$$

where the risk-neutral probability of default is the probability implied by current prices determined in markets with risk-neutral investors (who do not require an extra risk premium for the systematic risk of a credit-risky investment, although they are concerned about credit risk due to its effect on potential loss of principal).

ADVANTAGES AND DISADVANTAGES OF THE STRUCTURAL MODEL

The structural model to pricing risky debt has two key advantages.

1. Use of equity market data
   - The model uses equity market data, which is considered to be more reliable than bond market data due to the greater liquidity and transparency of equity markets compared to bond markets.
2. Different securities
   - It can readily deal with an issuer's different securities (e.g., bonds with different seniorities and convertible bonds), since the securities or tranches have the same assets and parameters.

The structural model has three key disadvantages.

1. Inaccurate estimates
   - Some of the model's parameters need to be estimated since they are not readily observable (e.g., market value of assets and return volatility). However, if equity prices are unreliable (e.g., based on illiquid data), then estimates of asset volatility and values will be inaccurate.
   - The Merton model is not successful at explaining the credit spread on short-term securities (i.e., using model parameters that correspond to historical data, the model predicts a very low credit spread for short-term assets, which contradicts empirical evidence).
2. Unreliable data
   - Current data on a firm's or structure's liabilities may be unreliable. In the case of sovereign debt, the data may be unusable.
3. Unreasonable valuation
   - Simple structural models sometimes generate unreasonable valuations, especially for short-term, high-quality debt and near-default debt.

L.O. 5.1.6

DEMONSTRATE KNOWLEDGE OF THE MERTON MODEL

The best known structural credit risk model is the Merton model.

Mechanics of Merton's structural model

The value of a levered firm equals the sum of its equity value and its risky debt value:

$$\text{Assets} = \text{Equity} + \text{Risky debt.}$$

Substituting the call option view of the firm's equity (i.e., Equity = Call option on firm's assets) and the put option view of its risky debt (i.e., Debt = Riskless bond − Put on firm's assets) into this equation gives:

$$\text{Assets} = \text{Call} + (\text{Riskless bond} - \text{Put}). \qquad (1)^{7}$$

- As discussed, the value of the firm's risky debt is the value of an otherwise identical riskless bond reduced by the put value, where the reduction in the debt value is the market's price for bearing the firm's credit risk.

[7] Equation (1) is the put-call parity relationship.

- Consider that the volatility of the firm's assets increases, but the current value of the firm's assets remains the same. All else equal, when the volatility of the firm's assets increases, the value of the firm's equity (like any call option) increases.
  - Equation (1) indicates that, for the value of the assets to remain the same, every dollar increase in the firm's equity value (i.e., the call) must result in a $1 decline in the risky debt value; and this decline is reflected in Equation (1) as an increase in the value of the put. This illustrates the conflict of interest between stockholders and bondholders.
  - The increase in equity value when volatility increases is due to equity's long vega exposure, and the decreases in debt value is due to debt's short vega exposure.

APPLYING MERTON'S MODEL TO VALUE DEBT AND PUTS ON A FIRM'S ASSETS

Example 1

A company has a zero-coupon bond with a face value of $72 million and a maturity date in one year. The company has $130 million in assets and $70 million in equity value. A one-year riskless zero-coupon bond is currently selling for 90% of its face value.

a) What is the value of the company's debt?

b) What is the value of a one-year put option on the company's assets with a strike price of $72 million?

a) Assets = Equity + Risky debt

$130m = $70m + Risky debt => Risky debt = $130m – $70m = $60m

b) Risky debt = Riskless bond − Put

$60m = (90% × $72m) – Put

$60m = $64.80m – Put => Put = $64.80m – $60m = $4.8m

VALUING RISKY DEBT USING BLACK-SCHOLES OPTION PRICING MODEL

Risky debt can be valued using the Black-Scholes option pricing model with put-call parity. The process involves four steps process.

1. Estimate the firm's equity volatility.
   - This is accomplished by analyzing the firm's historical stock volatility, the implied volatility of options on the firm's stock, or using a combination of historical and implied volatilities.
2. Unlever the firm's equity volatility (from step 1) based on its capital structure.
   - The firm's estimated asset volatility, σ_{assets}, is given by:

$$\sigma_{\text{assets}} \approx \sigma_{\text{equity}}\left(\frac{\text{Equity}}{\text{Assets}}\right),$$

where σ_{equity} represents the firm's estimated equity volatility.

3. Determine the price of a call and put option on the firm's assets.
   - Call and put prices are found using an option pricing model (e.g., the Black-Scholes) with the estimated asset volatility and values of observable variables (e.g., risk-free rate).
4. Value the firm's risky debt.
   - The risky debt price equals the price of a riskless bond less the price of the put.
   - The value of the firm's stock corresponds to the price of the call option.

The accuracy of the estimated option values depends on adherence to the assumptions of the option pricing model. Adherence to three assumptions may be particularly challenging.

1. Percentage changes in the firm's asset values over time are lognormally distributed.
2. The volatility of the firm's assets can be accurately estimated.
3. The firm has one zero-coupon debt instrument.

Example 2

A firm with $200 million in assets and $110 million in equity value has one-year debt with a face value of $100 million. The firm's equity volatility is estimated to be 36%. How can the value of the firm's equity be estimated given that the volatility of the firm's assets doubles?

$$\sigma_{\text{assets}} \approx \sigma_{\text{equity}}\left(\frac{\text{Equity}}{\text{Assets}}\right) = 36\%\left(\frac{\$110\text{m}}{\$200\text{m}}\right) = 19.8\%$$

The equity volatility is unlevered from 36% to an asset volatility of 19.8%.

Doubling the asset volatility increases it to 39.6%.

The value of the firm's equity corresponds to the price of a call on the firm's assets. The call option price may be found using an option pricing model and the following values: asset value of $200 million, asset volatility of 39.6%, strike price of $100 million, time to expiration of one year, and the prevailing risk-free rate.

MERTON & BLACK-SCHOLES MODELS – PRICING EQUITY AS A CALL OPTION

The Black-Scholes option pricing model may be used in the Merton model to value a firm's equity as a call option on the firm's assets. According to Black-Scholes, the value of a European call option may be expressed as:

$$E_t = A_t N(d) - Ke^{-r\tau} N\left(d - \sigma_A \sqrt{\tau}\right), \qquad \textbf{(G)}$$

where r is the annualized continually compounded short-term interest rate on risk-free debt, $\tau = T - t$ is the time left to maturity of the debt, σ_A is the annualized volatility of the return on the firm's assets,

$$d = \frac{\ln(A_t/K) + \left(r + 0.5\sigma_A^2\right)\tau}{\sigma_A \sqrt{\tau}}, \qquad \textbf{(G)}$$

and $N(\bullet)$ is the cumulative probability distribution function for a standard normal distribution [i.e., $N(d) = \Pr(Z \leq d)$, where Z is a standard normal random variable].

- In the event of default, equity holders lose at most their equity stake; however, bondholders face a potential loss of the shortfall between the asset value A_T and the debt's face value K. The bondholders could hypothetically hedge against this loss by buying a put option with strike K that pays out if asset value A_T falls below K. In this case, at any time t, the debt combined with the put guarantees a payoff of K at maturity time T.

USING BLACK-SCHOLES IN THE MERTON MODEL

Merton's model describes risky debt as a portfolio with a long position in risk-free debt (with the same maturity as the risky debt) and a short position in a put with the same maturity date as the debt and a strike price equal to the debt's face value K. Thus, the time-t value of risky debt may be expressed as:

$$D_t = Ke^{-r\tau} - \text{Put value}_t \cdot$$

This expression indicates that $\text{Put value}_t = Ke^{-r\tau} - D_t$ (i.e., Risk-free debt value – Risky debt value).

The Black-Scholes option pricing model can be used to price the put component of the value of debt. According to Black-Scholes, the value of a European put may be expressed as:

$$P_t = Ke^{-r\tau} \times N\left(-d + \sigma_A \sqrt{\tau}\right) - A_t \times N(-d) \cdot \qquad \textbf{(G)}$$

USING CREDIT SPREAD TO CALCULATE PRICE OF RISKY DEBT

The value of risky debt is often determined by accounting for the debt's credit risk in terms of a spread above the risk-free rate. Based on this, the value of zero-coupon debt may be expressed as:

$$D_t = Ke^{-(r+s_t)\tau},$$

where s_t is the annual spread (expressed as a continuously compounded rate) due to credit risk. The credit spread s may be expressed in terms of the variables in the Merton model as:

$$s_t = -\frac{1}{\tau} \times \ln\left[N\left(d - \sigma_A \sqrt{\tau}\right) + \frac{A_t}{K} e^{r\tau} \times N(-d)\right]. \qquad \textbf{(G)}$$

Thus, the Merton model has two key outputs: the probability of default and the credit spread.

Example 3

The ABC Corporation has $200 million worth of assets and 3-year zero-coupon debt with a face value of $160 million. The volatility of the firm's assets is 20% and the annual risk-free rate is 5%. $N(1.2504) = 0.8944$ and $N(0.904) = 0.817$. Under Merton's model:

a) What is the current value of ABC's equity and debt?

b) What is the value of the put option given to ABC's equity holders?

c) What is the credit spread on ABC's debt?

a) $E_t = A_t N(d) - Ke^{-r\tau} N\left(d - \sigma_A \sqrt{\tau}\right)$

$$= \$200\text{m} \times N(d) - \$160\text{m} \times e^{-0.05\times3} \times N\left(d - 0.2\sqrt{3}\right)$$

Find d: $d = \dfrac{\ln\left(\dfrac{200}{160}\right) + \left(0.05 + \dfrac{0.2^2}{2}\right)\times 3}{0.2\times\sqrt{3}} = 1.2504$.

Find the $N(\bullet)$s:

$$N(d) = N(1.2504) = 0.8944$$

$$N\left(d - 0.2\sqrt{3}\right) = N\left(1.2504 - 0.2\sqrt{3}\right) = N(0.904) = 0.817$$

Now, substitute these into E_t:

$$E_t = \$200\text{m} \times N(d) - \$160\text{m} \times e^{-0.05\times3} \times N\left(d - 0.2\sqrt{3}\right)$$
$$= \$200\text{m} \times 0.8944 - \$160\text{m} \times e^{-0.05\times3} \times 0.8170 = \$66.37\text{m}$$

Assets A_t = Debt D_t + Equity E_t

$200m = D_t + $66.37m $\Rightarrow$ $D_t = \$200\text{m} - \$66.37\text{m} = \$133.63\text{m}$

b) $\text{Put price}_t = Ke^{-r\tau} - D_t$ (i.e., Risk-free debt price − Risky debt price)

$$= \$160e^{-0.05\times3} - \$133.63 = \$4.08m$$

The put option price can also be found using the Black-Scholes model as:

$$P_t = Ke^{-r\tau} \times N\left(-d + \sigma_A\sqrt{\tau}\right) - A_t \times N(-d)$$
$$= \$160\text{m} \times e^{-0.05\times3} \times N\left(-1.2504 + 0.2\sqrt{3}\right) - \$200\text{m} \times N(-1.2504)$$
$$= \$160\text{m} \times e^{-0.05\times3} \times N(-0.904) - \$200\text{m} \times N(-1.2504)$$
$$= \$160\text{m} \times e^{-0.05\times3} \times (1 - 0.817) - \$200\text{m} \times (1 - 0.8944) = \$4.08\text{m}$$

Note: $N(-d) = 1 - N(d)$. For instance, if $N(1.2504) = 0.8944$, then $N(-1.2504) = 1 - 0.8944$.

c) Credit spread: $s_t = -\frac{1}{\tau} \times \ln\left[N\left(d - \sigma_A\sqrt{\tau}\right) + \frac{A_t}{K}e^{r\tau} \times N(-d)\right]$

$$= -\frac{1}{\tau} \times \ln\left[N(0.904) + \frac{A_t}{K}e^{r\tau} \times N(-1.2504)\right]$$

$$= -\frac{1}{3} \times \ln\left[0.817 + \frac{\$200\text{m}}{\$160\text{m}}e^{0.05\times3} \times (1-0.8944)\right] = 1.003\%$$

The spread may also be found by expressing the face value of the debt ($160m) as the current debt value ($133.63m) compounded forward at 5% for three years; as shown below.

$$\text{Face value of debt} = \text{Current value of debt} \times e^{(r+s)t}$$

$$\$160\text{m} = \$133.63\text{m} \times e^{(0.05+s)\times3}$$

$$\frac{\$160\text{m}}{\$133.63\text{m}} = e^{(0.05+s)\times3}$$

$$1.1973 = e^{(0.05+s)\times3}$$

$$\ln 1.1973 = (0.05+s)\times3$$

$$(0.05+s) = \frac{\ln 1.1973}{3} = \frac{0.1801}{3} = 0.06003$$

$$s = 0.06003 - 0.05 = 0.01003 = 1.003\%$$

MERTON MODEL – FOUR IMPORTANT PROPERTIES

The Merton model has four important properties related to the sensitivity of its two key outputs (i.e., probability of default and credit spread) to changes in four of its inputs (i.e., the debt's maturity, the assets' volatility, the firm's leverage, and the riskless interest rate).

1. Sensitivity to maturity
   - As the debt's time to maturity increases, the cumulative probability of default increases. This occurs because, as the time to maturity increases, a firm's asset value has more chances to decline below the face value of debt. In contrast, the non-cumulative probability of default may increase or decrease over time.
   - Sensitivity of credit spreads to maturity may be determined by examining the shape of the term structure of credit spreads for different levels of leverage, asset volatility, and riskless rate (since the term structure shape is driven by the behavior of the default probability over time for a given level of leverage, volatility, and riskless rate). This discussion focuses on the sensitivity of credit spreads to maturity over different leverage levels.
   - Figure 2 depicts credit spreads over different bond maturities for three levels of leverage (i.e., debt/assets): 10%, 30%, and 60%; with 30% volatility and 5% risk-free rate. As indicated, the shape of the term structure of credit spreads varies depending on the level of leverage, from uniformly upward sloping for firms with very low leverage (Graph A) to humped curves for firms with higher leverage (Graph C).
     - Graph A (low 10% leverage): bonds with short maturities have a low credit spread and thus a low probability of default. This is because, in short time periods, the probability that the firm's assets will decline below the debt's face value is low. However, bonds