FRM Part II Exam

By AnalystPrep

Study Notes - Market Risk Measurement and Management

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Reading 63: Estimating Market Risk Measures: An Introduction and Overview

After completing this reading, you should be able to:

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- Estimate the expected shortfall given P/L or return data.
- Describe coherent risk measures.
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Estimating VaR Using a Historical Simulation Approach

One of the simplest approaches to estimating VaR involves historical simulation. In this case, a risk manager constructs a distribution of losses by subjecting the current portfolio to the actual changes in the key factors experienced over the last t time periods. After that, the mark-to-market P/L amounts for each period are calculated and recorded in an ordered fashion.

Let's assume that we have a list of 100 ordered P/L observations and we would like to determine the VaR at 95% confidence. This implies that we should have a 5% tail, and that there are 5 observations (= 5% × 100) in the tail. In this scenario, the 95% VaR would be the sixth largest P/L observation.

In general, if there are n ordered observations, and a confidence level cl%, the cl% VaR is given by the $[(1-cl\%)n+1]^{th}$ highest observation. This is the observation that separates the tail from the body of the distribution. For instance, if we have 1,000 observations and a confidence level of 95%, the 95% VaR is given by the $(1-0.95)1,000+1=51^{st}$ observation. There are 50 observations in the tail.

Example: Computing VaR Using the Historical Simulation Approach

Over the past 300 trading days, the five worst daily losses (in millions) were: -30, -27, -23, -21, and -19. If the historical window is made up of these 300 daily P&L observations, what is the 99% daily HS VaR?

Solution

VaR is to be estimated at 99% confidence. This means that 1% (i.e., 3) of the ordered observations would fall in the tail of the distribution.

The 99% VaR would be given by the (1 – 0.99) 300 + 1 highest observation, i.e., the 4^{th} highest value. This would be -21.

Note that the 4^{th} highest observation would separate the 1% of the largest losses from the remaining 99% of returns.

Estimating Parametric VaR

When estimating VaR using the historical simulation approach, we do not make any assumption regarding the distribution of returns. In contrast, the parametric approach explicitly assumes a distribution for the underlying observations. We shall be looking at (1) VaR for returns that follow the normal distribution and (II) VaR for returns that follow the lognormal distribution.

I. Normal VaR

a. Profit/Loss Data

We have already established that the VaR for a given confidence level indicates the point that separates tail losses from the body of the distribution. Using P/L data, our VaR is:

$$\text{VaR}(\alpha\%) = -\mu_{PL} + \sigma_{PL} \times z_{\alpha}$$

where $\mu_{PL}$and σ_{PL} are the mean and standard deviation of P/L, and z_{α} is the standard normal variate corresponding to the chosen confidence level. If we take the confidence level to be cl, then z_{α} is the standard normal variate such that cl of the probability density function lies to the left while 1-cl of the probability density lies to the right. In most cases, you will be required to calculate the VaR when cl = 95%, in which case the standard normal variate is -1.645.

Notably, the VaR cutoff will be on the left side of the distribution. As such, the VaR is usually negative but is reported as positive since it is the value that is at risk (the negative amount is implied).

Example: Computing VaR (Normal Distribution)

Let P/L for ABC limited over a specified period be normally distributed with a mean of $12 million and a standard deviation of $24 million. Calculate the 95% VaR and the corresponding 99% VaR.

Solution

$$\begin{aligned} 95\%\ \text{VaR}: &\quad \alpha = 95 \\ \text{VaR}(95\%) &= -\mu_{PL} + \sigma_{PL} \times z_{95} \\ &= -12 + 24 \times 1.645 = 27.48 \end{aligned}$$

How do we interpret this? ABC expects to lose at most $27.48 million over the next year with 95% confidence. Equivalently, ABC expects to lose more than $27.48 million with a 5% probability.

$$\begin{aligned} 99\%\ \text{VaR}: &\quad \alpha = 99 \\ \text{VaR}(99\%) &= -\mu_{PL} + \sigma_{PL} \times z_{99} \\ &= -12 + 24 \times 2.33 = 43.824 \end{aligned}$$

The 99% VaR can be interpreted in a similar fashion. However, note that the VaR at 99% confidence is significantly higher than the VaR at 95% confidence. Generally, the VaR increases as the confidence level increases.

b. Arithmetic Data

When using arithmetic data rather than P/L data, VaR calculation follows a similar format.

Assuming the arithmetic returns follow a normal distribution,

$$r_t = \frac{p_t + D_t - p_{t-1}}{p_{t-1}}$$

Where p_t: Asset price at the end of periods; D_t: Interim payments

The VaR is:

$$\text{VaR}(\alpha\%) = [-\mu_r + \sigma_r \times z_\alpha]p_{(t-1)}$$

Example 1 : Computing VaR given arithmetic data

The arithmetic returns r_t, over some period of time, are normally distributed with a mean of 1.34 and a standard deviation 1.96. The portfolio is currently worth $1 million. Calculate the 95% VaR and 99% VaR.

Solution

$$\begin{aligned}
\text{VaR}(\alpha\%) &= [-\mu_r + \sigma_r \times z_\alpha]p_{t-1} \\
95\%\text{VaR}: &\quad \alpha = 95 \\
\text{VaR}(95\%) &= [-1.34 + 1.96 \times 1.645] \times \$1 = \$1.8842 \text{ million} \\
99\%\ \text{VaR}: &\quad \alpha = 99 \\
\text{VaR}(99\%) &= [-1.34 + 1.96 \times 2.33] \times \$1 = \$3.2190 \text{ million}
\end{aligned}$$

Again, note that as the confidence level increases, so does the VaR

Example 2: Computing VaR Given Arithmetic Data

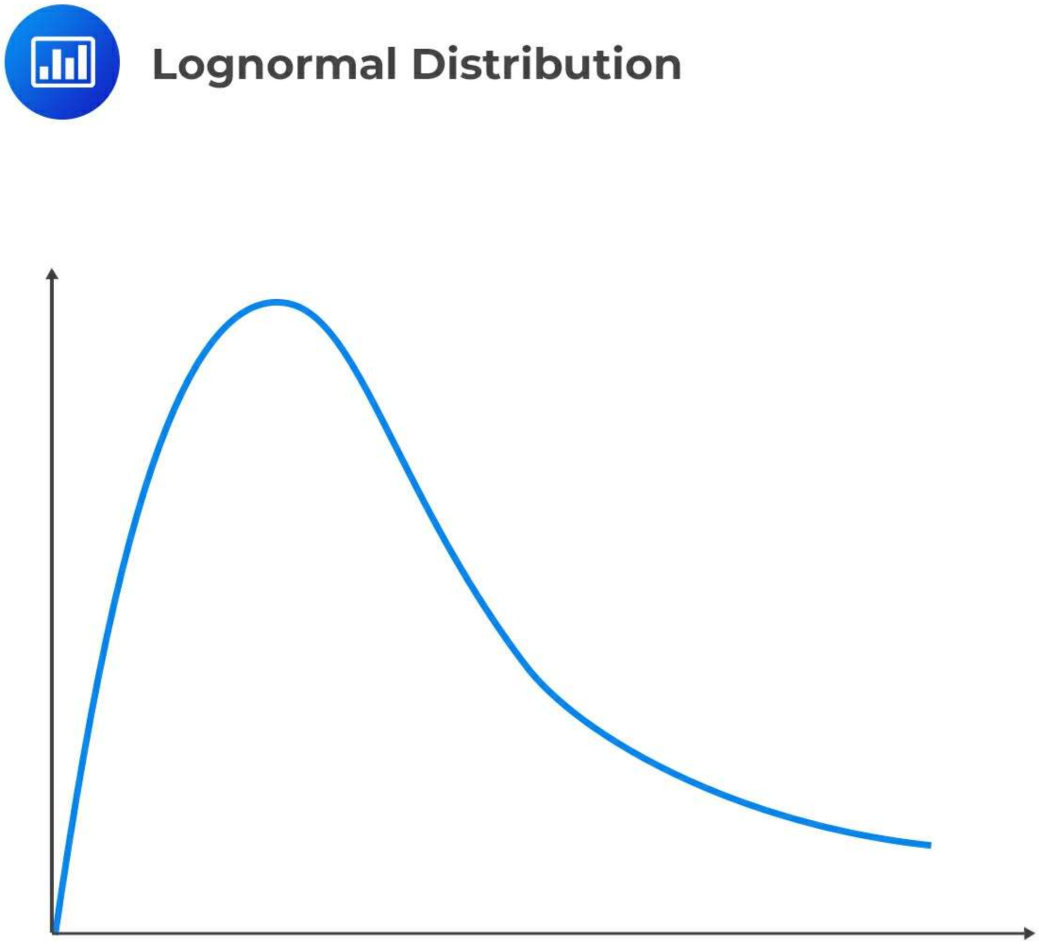
A portfolio has a beginning period value of $200. The arithmetic returns follow a normal distribution with a mean of 15% and a standard deviation of 20%. Determine the VaR at both the 95% and 99% confidence levels.

Solution

$$\text{VaR}(\alpha\%) = [-\mu_r + \sigma_r \times z_\alpha]p_{t-1}$$
$$95\%\ \text{VaR}: \quad \alpha = 95$$
$$\text{VaR}(95\%) = [-0.15 + 0.2 \times 1.645] \times \$200 = \$35.8 \text{ million}$$
$$99\%\ \text{VaR}: \quad \alpha = 99$$
$$\text{VaR}(99\%) = [-0.15 + 0.2 \times 2.33] \times \$200 = \$63.2 \text{ million}$$

II. Lognormal VaR

Unlike the normal distribution, the lognormal distribution is bounded by zero. Besides, it is also skewed to the right.



This explains why this is the favored distribution when modeling the prices of assets such as stocks (which can never be negative). We also ditch the arithmetic returns in favor of geometric ones. As earlier established, the geometric return is:

$$R_t = \ln\left[\frac{p_t + D_t}{p_{t-1}}\right]$$

If we assume that geometric returns follow a normal distribution (μ_R, σ_R), then the natural logarithm of asset prices follows a normal distribution and p_t follows a lognormal distribution

It can be shown that:

$$VaR(\alpha\%) = (1 - e^{\mu_R - \sigma_R \times z_\alpha})p_{t-1}$$

Example: Computing Lognormal VaR

Assume that geometric returns over a given period are distributed as normal with a mean of 0.1 and standard deviation of 0.15, and we have a portfolio currently valued at $20 million. Calculate the VaR at both 95% and 99% confidence.

Solution

$$\begin{aligned} &VaR(\alpha\%) = (1 - e^{\mu_R - \sigma_R \times z_\alpha})p_{(t-1)} \\ &95\%\ VaR: \quad \alpha = 95 \\ &VaR(95\%) = (1 - e^{0.1-0.15\times1.645})20 = \$2.7298 \text{ million} \\ &99\%\ VaR: \quad \alpha = 99 \\ &VaR(99\%) = (1 - e^{0.1-0.15\times2.33})20 = \$4.4162 \text{ million} \end{aligned}$$

Example 2: Computing Lognormal VaR

Let's assume that the geometric returns Rt, are distributed as normal with a mean 0.06 and standard a deviation 0.30. The portfolio is currently worth $1 million. Calculate the 95% and 99% lognormal VaR.

Solution

$$\begin{aligned} &VaR(\alpha\%) = (1 - e^{\mu_R - \sigma_R \times z_\alpha})p_{(t-1)} \\ &95\%\ VaR: \quad \alpha = 95 \\ &VaR(95\%) = (1 - e^{0.06-0.30\times1.645})1 = \$0.3518 \text{ million} \\ &99\%\ VaR: \quad \alpha = 99 \\ &VaR(99\%) = (1 - e^{0.06-0.30\times2.33})1 = \$0.4689 \text{ million} \end{aligned}$$

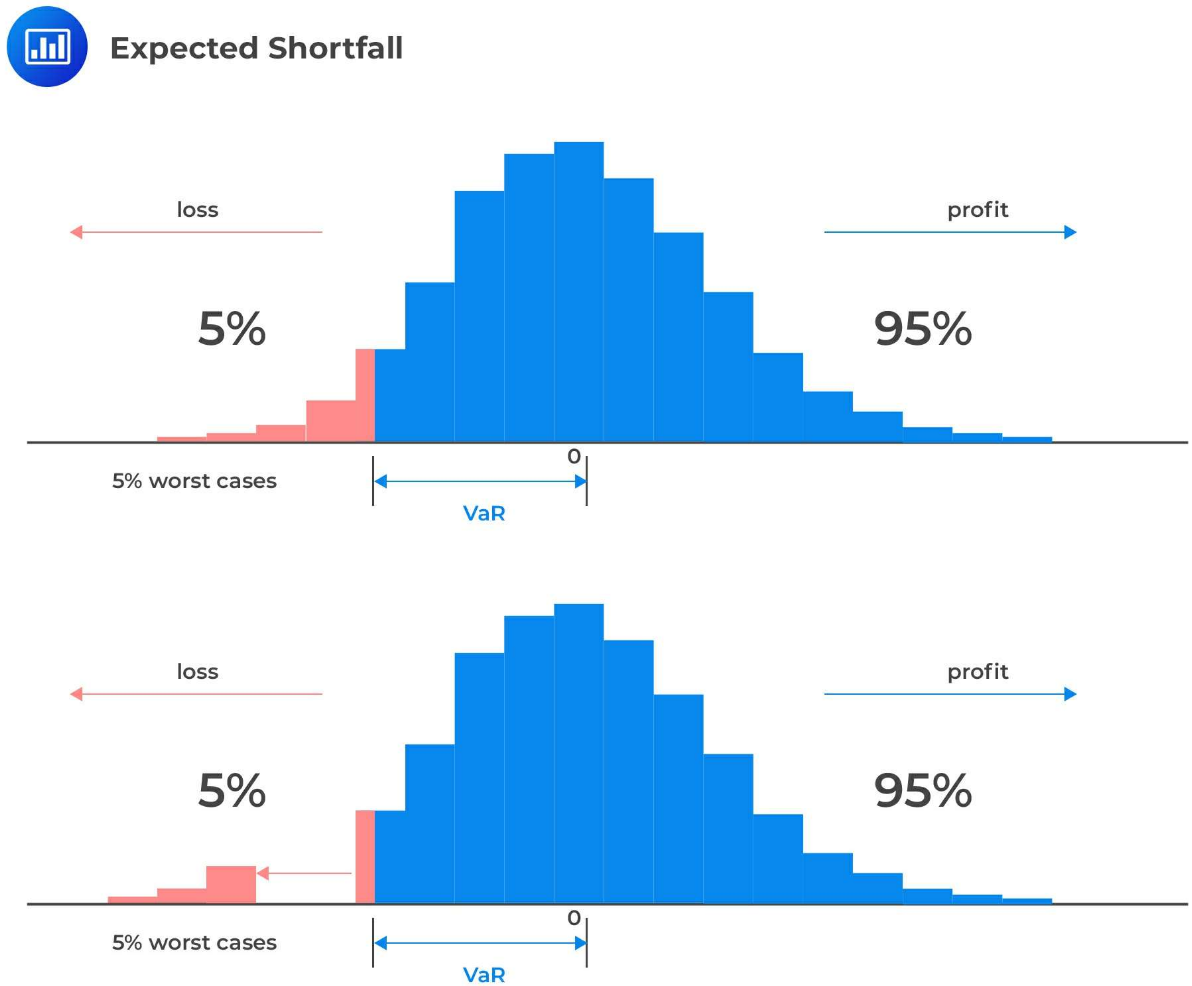
Estimating the Expected Shortfall Given P/L or Return Data

Despite the significant role VaR plays in risk management, it stops short of telling us the amount or magnitude of the actual loss. What it tells us is the maximum value we stand to lose for a given confidence level. If the 95% VaR is, say, $2 million, we would expect to lose not more than $2 million with 95% confidence. However, we do not know what amount the actual loss would be.

To have an idea of the magnitude of the expected loss, we need to compute the expected shortfall.

Expected shortfall (ES) is the expected loss given that the portfolio return already lies below the pre-specified worst-case quantile return, e.g., below the 5th percentile return. Put differently, expected shortfall is the mean percent loss among the returns found below the q-quantile (q is usually 5%). It helps answer the Practice Question: If we experience a catastrophic event, what is the expected loss in our financial position?

The expected shortfall (ES) provides an estimate of the tail loss by averaging the VaRs for increasing confidence levels in the tail. It is also called the expected tail loss (ETL) or the conditional VaR.



To determine the ES, the tail mass is divided into n equal slices and the corresponding n - 1 VaRs are computed.

Illustration

Assume that we wish to estimate a 95% ES on the assumption that losses are normally distributed with a mean of 0 and a standard deviation of 1. Ideally, we would need a large value of n to improve accuracy and reliability and then use the appropriate computer software. For illustration purposes, however, let's assume that n = 10.

This value gives us 9 tail VaRs (i.e., 10 – 1) at confidence levels in excess of 95%. These vaRs are listed below. The estimated ES is the average of these VaRs.

Confidence level	Tail VaR
95.5%	1.6954
96.0%	1.7507
96.5%	1.8119
97.0%	1.8808
97.5%	1.9600
98.0%	2.0537
98.5%	2.1701
99.0%	2.3263
99.5%	2.5738
	Average of tail VaRs = ES = 2.0250

The theoretical true value of the ES can be worked out by using different values of n. This can easily be achieved with the help of computer software.

Number of tail slices (n)	ES
10	2.0250
25	2.0433
50	2.0513
100	2.0562
250	2.0597
500	2.0610
1,000	2.0618
2,500	2.0623
5,000	2.0625
10,000	2.0626
	Theoretical average = 2.063

These results show that the estimated ES rises with n and gradually converges at the theoretical true value of 2.063.

Example: Computing the Expected Shortfall Given P/L Data

A market risk manager uses historical profit/loss data from 300 days to calculate a 95% Value at Risk (VaR) of CAD30 million. Loss observations strictly exceeding this threshold (in millions) are as follows: CAD31, 32, 33, 35, 37, 38, 39, 41, 42, 43, 45, 46, 47, 48, and 50. What is the conditional VaR (expected shortfall) based on these tail losses?

Solution

Expected shortfall is the average of tail losses.

$$\text{ES} = \left(\frac{31+32+33+35+37+38+39+41+42+43+45+46+47+48+50}{15}\right)$$
$$= \text{CAD } 40.47 \text{ million}$$

Coherent Risk Measures

If X and Y are the future values of two risky positions, a risk measure $\rho(\bullet)$ is said to be coherent if it satisfies the following properties:

I. **Sub-additivity**: $\rho(X+Y) \leq \rho(X) + \rho(Y)$

   Interpretation: If we add two portfolios to the total risk, the risk measure can't get any worse than adding the two risks separately.

II. **Homogeneity**: $\rho(XX) = X\rho(X)$

   Interpretation: Doubling a portfolio consequently doubles the risk.

III. **Monotonicity**: $\rho(X) \geq \rho(Y),\ \text{if } X \leq Y$

   Interpretation: If one portfolio has better values than another under all scenarios then its risk will be better.

IV. **Translation invariance**: $\rho(X+n) = \rho(X) - n$

   Interpretation: the addition of a sure amount n (cash) to our position will decrease our risk by the same amount because it will increase the value of our end-of-period portfolio.

VaR is not a coherent risk measure because it fails to satisfy the sub-additivity property. The expected shortfall (ES), however, does satisfy this property and is, therefore, a coherent risk measure. If we combine two portfolios, the total ES would usually decrease to reflect the benefits of diversification. The ES would certainly never increase because it takes correlations into account. By contrast, the total VAR can – and in fact occasionally does – increase. As a result, the

ES does not discourage risk diversification, but the VaR sometimes does.

The ES tells us what to expect in bad (i.e., tail) states—it gives an idea of how bad it might be, whilst the VaR tells us nothing other than to expect a loss higher than the VaR itself.

The ES is also better justified than VAR in terms of decision theory.

Assume that you are faced with a choice between two portfolios, A and B, with different distributions. Under first − order stochastic dominance (FSD) which is a rather strict decision rule, A has first-order stochastic dominance over random variable B if, for any outcome x, A gives at least as high a probability of receiving at least x as does B, and for some x, A gives a higher probability of receiving at least x. And under second − order stochastic dominance (SSD), a more realistic decision rule, portfolio A would dominate B if it has a higher mean and lower risk. Using the

ES as a risk measure is consistent with SSD, whereas VAR requires FSD, which is less realistic.

Overall, the ES dominates the VaR and presents a stronger case for use in risk management.

Estimating Risk Measures by Estimating Quantiles

It is possible to estimate coherent risk measures by manipulating the "average VaR" method. A coherent risk measure is a weighted average of the quantiles (denoted by q_p of the loss distribution):

$$M_{\emptyset} = \int_0^1 \emptyset(p)\, q_p\, dp$$

where the weighting function $\emptyset(p)$ is specified by the user, depending on their risk aversion. The ES gives all tail-loss quantiles an equal weight of [1/(1 – cl)] and other quantiles a weight of 0. Therefore, the ES is a special case of $M_{\emptyset}$.

Under the more general coherent risk measure, the entire distribution is divided into equal probability slices weighted by the more general risk aversion (weighting) function.

We could illustrate this procedure for n = 10. The first step is to divide the entire return distribution into nine (10 – 1) equal probability mass slices (loss quantiles) as shown below. Each breakpoint indicates a different quantile.

For example, the 10% quantile(confidence level = 10%) relates to -1.2816, the 30% quantile (confidence level = 30%) relates to -0.5244, the 50% quantile (confidence level = 50%) relates to 0.0,and the 90% quantile (confidence level = 90%) relates to 1.2816. After that, each quantile is weighted by the specific risk aversion function and then averaged to arrive at the value of the coherent risk measure.

Confidence Level	Normal Deviate (A)	Weight (B)	A × B
10%	−1.2816	0	
20%	−0.8416	0	
30%	−0.5244	0	
40%	−0.2533	x	
50%	0.0	xx	
60%	0.2533	xxx	
70%	0.5244	xxxx	
80%	0.8416	xxxxx	
90%	1.2816	xxxxxx	
			Average risk measure = sum of A × B

The xs in the third column represent weight that depends on the investor's risk aversion.

Compared to the expected shortfall, such a coherent risk measure is more sensitive to the choice of n. However, as n increases, the risk measure converges at its true value. Remember that increasing the value of n takes us farther into some very extreme values at the tail.

Evaluating Estimators of Risk Measures by Estimating Their Standard Errors

Bear in mind that any risk measure estimates are only as useful as their precision. The true value of any risk measure is unknown and therefore, it is important to come up with estimates in a precise manner. Why? Only then can we be confident that the true value is fairly close to the estimate. Hence, it is important to supplement risk measure estimates with some indicator that gauges their precision. The standard error is a useful indicator of precision. More generally, confidence intervals (built with the help of standard errors) can be used.

The big Practice Question is: How do we go about determining standard errors and establishing confidence intervals? Let’s start with a sample size of n and arbitrary bin width of h around quantile, q. Bin width refers to the width of the intervals, or what we usually call “bins,” in a (statistical) histogram. The square root of the variance of the quantile is equal to the standard error of the quantile. Once the standard error has been specified, a confidence interval for a risk measure can be constructed:

$$[q + se(q) \times z_\alpha] > VaR > [q - se(q) \times z_\alpha]$$

Example: Computing the Standard Error for a Risk Measure

Construct a 90% confidence interval for 5% VaR (the 95% quantile) drawn from a standard normal distribution. Assume bin width = 0.1 and that the sample size is equal to 1,000.

Solution

Step 1: Determine the value of q.

The quantile value, q, corresponds to the 5% VaR. For the normal distribution, the 5% VaR occurs at 1.645 (implying that q = 1.645). So in crude form, the confidence interval will take the following shape:

$$[q + se(q) \times z_\alpha] > VaR > [q - se(q) \times z_\alpha]$$

Step 2: Determine the range of q.

For the bin width of 0.1, we know that q falls in the bin spanning 1.645 ± 0.1/2 = [1.595,1.695].

Note: The left tail probability, p, is the area to the left of 1.695 for a standard normal distribution.

Step 3: Determine the probability mass f(q).

We wish to calculate the probability mass between 1.595 and 1.695, represented as f(q). From the normal distribution table, the probability of a loss exceeding 1.695 is 4.5% (which is also equal to p) and the probability of profit or a loss less than 1.595 is 94.46%. Hence, f(q) = 1 – 0.045 – 0.9446 = 1.032%.

Step 4: Calculate the standard error of the quantile from the variance approximation of q.

$$se(q) = \frac{\sqrt{\frac{p(1-p)}{n}}}{f(q)}$$

In this case,

$$se(q) = \frac{\sqrt{\frac{0.045 \times 0.955}{1000}}}{0.01032} = 0.63523$$

Therefore, the following gives us the required CI:

$$[1.645 + 0.63523 \times 1.645] > VaR > [1.645 - 0.63523 \times 1.645]$$
$$= 2.69 > VaR > 0.6$$

Important:

Unlike the VaR, confidence intervals are two-sided. So, for a 90% CI, there will be 5% in each tail. This is equivalent to the 5% significance level of VaR, and as such, the critical values are ± 1.645.

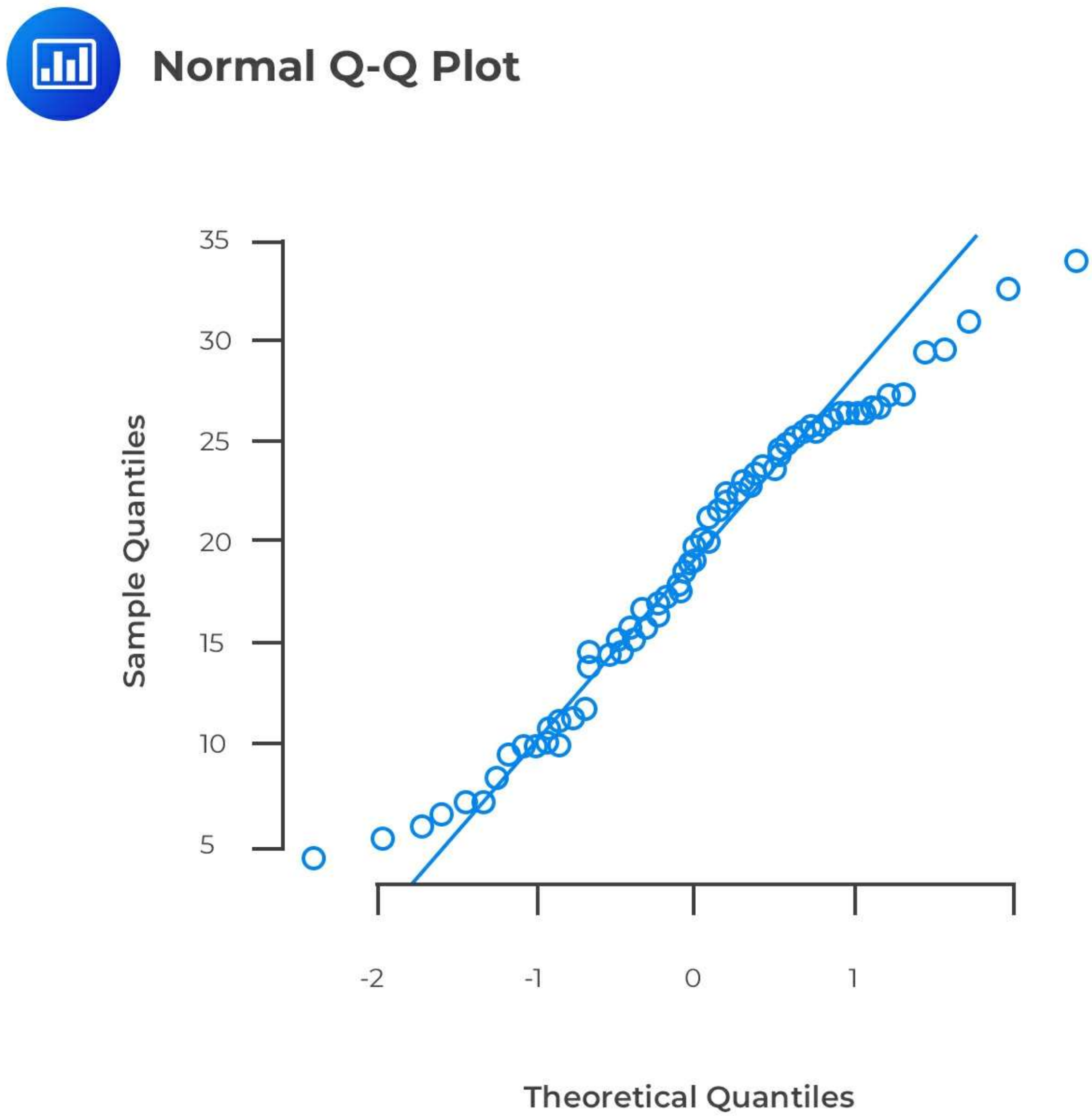
The larger the sample size, the smaller the standard error, and the narrower the confidence interval.

Increasing the bin size, h, holding all else constant, will increase the probability mass f(q) and reduce p, the probability in the left tail. Subsequently, the standard error will decrease and the confidence interval will again narrow.

Increasing p implies that tail probabilities are more likely. When that happens, the estimator becomes less precise, and standard errors increase, widening the confidence interval. Note that the expression p(1- p) will be maximized at p = 0.5.

Interpreting QQ Plots to Identify the Characteristics of a Distribution

The quantile-quantile plot, more commonly called the Q-Q plot, is a graphical tool we can use to assess if a set of data plausibly came from some theoretical distributions such as a Normal or exponential.



For example, if we conduct risk analysis assuming that the underlying data is normally distributed, we can use a normal QQ plot to check whether that assumption is valid. We would need to plot the quantiles of our data set against the quantiles of the normal distribution. It’s **not** a perfect air-tight verification modality but a visual proof that can be quite subjective.

Remember that by a quantile, we mean the fraction (or percentage) of points below a given value. For example, the 0.1 (or 10%) quantile is the point at which 10% percent of the data fall below and 90% fall above that value.

Why are QQ Plots Important?

To start with, we can use a QQ plot to form a tentative view of the distribution from which our data might be drawn. This involves specification of a variety of alternative distributions and construction of QQ plots for each. If the data are drawn from the reference population, then the QQ plot should be linear. Any reference distributions that produce non-linear QQ plots can then be dismissed, and any distribution that produces a linear QQ plot is a good candidate distribution for our data.

In addition, since a linear transformation in one of the distributions in a QQ plot changes the intercept and slope of the QQ plot we can use the intercept and slope of a linear QQ plot to give us a rough idea of the location and scale parameters of our sample data.

Ultimately, if the empirical distribution has heavier tails than the reference distribution, the QQ plot will have steeper slopes at its tails, even if the central mass of the empirical observations is approximately linear.

In conclusion, a QQ plot is good for identifying outliers (e.g., observations contaminated by large errors).

Illustration

The chart here shows a QQ plot for a data sample drawn from a normal distribution, compared to a reference distribution that is also normal. The central mass observations fit a linear QQ plot very closely while the observations at the tail are a bit spread out. In this case, the empirical distribution matches the reference population.