Chapter 1 Fundamentals of Probability

Version 1.0

Introduction

This chapter covers the fundamentals of probability:

- Conditional probability
- Independence and conditional independence
- Bayes' rule

1. Sample Space, Event Space, and Events

Fundamental Concepts

A *random variable* is an uncertain quantity/number. For example, when you roll a die, the result is a random variable.

An *outcome* is the observed value of a random variable. For example, if you roll a 2, it is an outcome.

An *event* can be a single outcome or a set of outcomes. For example, you can define an event as rolling a 2 or rolling an even number (i.e. getting a 2, 4 or 6).

Probability measures the likelihood of an event.

- An event with a probability of 0 never occurs.
- An event with a probability of 1 always occurs.
- Probabilities cannot be less than 0 or greater than 1.

For example, the probability of rolling a 2 is 1/6 = 16.7%. The probability of rolling an even number is 3/6 = 50%.

An *event space* is the set of all possible outcomes and combinations of outcomes. Event space is an abstract concept and does not have any specific application. For example, the event space for a coin toss is: heads, tails, both heads and tails, neither heads nor tails. Note that this event space also contains two impossible events: 'both heads and tails, and 'neither heads nor tails.'

Mutually Exclusive and Exhaustive events

Mutually exclusive events are events that cannot happen at the same time. For example, rolling a 2 and rolling a 3 are examples of mutually exclusive events. They cannot happen at the same time.

Exhaustive events are those that cover all possible outcomes. For example, 'rolling an even number' or' rolling an odd number' are exhaustive events. They cover all possible outcomes.

The sum of the probabilities of mutually exclusive and exhaustive events is equal to 1.

Conditional v/s Unconditional probabilities

Unconditional probability is the probability of an event occurring irrespective of the occurrence of other events. It is denoted as P(A). Unconditional probability is also called 'marginal' probability.

Conditional probability is the probability of an event occurring given that another event has occurred. It is denoted as P(A|B), which is the probability of event A given that event B has occurred.

For example, the probability of rolling a 3 on a fair die is 1/6. This is the unconditional probability of the event. However, if it is known that the number rolled is odd, then the probability of rolling a three is 1/3. This is the conditional probability of the event.

Joint Probability and Multiplication Rule

Multiplication rule is used to determine the joint probability of two events. The joint probability of two events A and B is expressed as:

$$P(AB) = P(A|B)\ P(B)$$

Example

P(interest rates will decrease) = P(D) = 40%
P(stock price increases) = P(S)
P(stock price will increase given interest rates decrease) = P(S|D) = 70%

Compute probability of a stock price increase **and** an interest rate decrease.

Solution:

$P(SD) = P(S|D) \times P(D) = 0.7 \times 0.4 = 0.28 = 28\%$

Rearranging the equation, we get the formula for computing conditional probabilities:

$$P(A|B) = P(AB) / P(B)$$

Addition Rule for Probabilities

Addition rule is used to determine the probability that at least one of the events will occur. It is expressed as:

$$P(A \text{ or } B) = P(A) + P(B) - P(AB)$$

P(AB) represents the joint probability that both A and B will occur. It is subtracted from the sum of the unconditional probabilities: P(A) + P(B), to avoid double counting.

If the two events are mutually exclusive, the joint probability: P(AB) is zero and the probability that either A or B will occur is simply the sum of the unconditional probabilities for each event:

$$P(A \text{ or } B) = P(A) + P(B)$$

Example

P(price of A increases) = P(A) = 0.5
P(price of B increases) = P(B) = 0.7
P(price of A and B increases) = P(AB) = 0.3

Compute the probability that the price of stock A **or** the price of stock B increases.

Solution

P(A or B) = 0.5 + 0.7 – 0.3 = 0.9

Total Probability Rule

The total probability rule is used to calculate the unconditional probability of an event, given conditional probabilities.

In investment analysis, we often formulate a set of mutually exclusive and exhaustive scenarios and then estimate the probability of a particular event. For example, let's say that we have two scenarios S and non-S that are mutually exclusive and exhaustive.

According to the total probability rule, the probability of any event P(A) can be expressed as:

$$P(A) = P(AS) + P(AS^C)$$

Using the multiplication rule we get,

$$P(A) = P(A|S)\,P(S) + P(A|S^C)\,P(S^C)$$

If we have more than two scenarios, we can generalize this equation to:

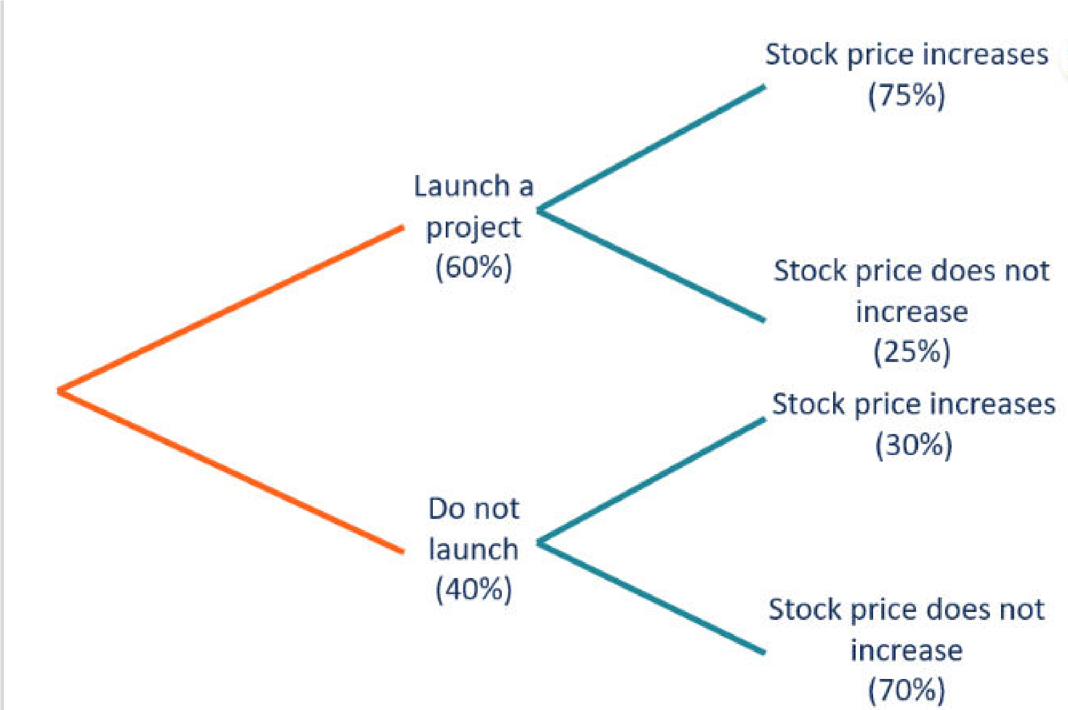
$$P(A) = P(AS_1) + P(AS_2) + \ldots + P(AS_n) = P(A|S_1)\,P(S_1) + P(A|S_2)\,P(S_2) + \ldots + P(A|S_n)\,P(S_n)$$

Example:

A stock analyst following a company discovers that the company is planning to launch a new project that is likely to affect the company's stock price. If the company launches the project, there is a 75% probability that its stock price will increase. If the company does not launch the project, there is a 30% probability that its stock price will increase. There is a 60% probability that the new project will be launched. Calculate the probability that the company's stock price will increase.

Solution:

We can use a decision tree to solve this problem.



Using the total probability rule:

$P(A) = P(A|S)\,P(S) + P(A|S^C)\,P(S^C)$

P(Stock price increase) = P(Stock price increase| New project launched) x P(New project launched) + P(Stock price increase| New project not launched) x P(New project not launched)

P(Stock price increase) = 0.75 x 0.6 + 0.30 x 0.4 = 0.57 = 57%

Thus, there is a 57% probability that the company's stock price will increase.

2. Independence

Independent and Dependent Events

If the occurrence of one event does not influence the occurrence of the other event, then the two events are called *independent events.*

i.e. $P(A|B) = P(A)$ or $P(B|A) = P(B)$

Multiplication rule for independent events: $P(AB) = P(A)\,P(B)$

Addition rule for independent events: $P(A \text{ or } B) = P(A) + P(B) - P(AB)$. (The addition rule does not change.)

If the probability of an event is affected by the occurrence of another event, then it is called a *dependent event.*

Consider rolling a die twice. Getting a 3 on the first roll does not change the probability of getting a 3 on the second roll. Therefore, the two events are independent. In this case, the joint probability of getting a 3 on both rolls is simply the product of their unconditional probabilities. The probability of getting a 3 on both rolls is 1/6 x 1/6 = 2.77%

Conditionally Independent Events

Two conditional probabilities, P(A|C) and P(B|C) may be independent or dependent regardless of whether the unconditional probabilities P(A) and P(B) are independent or not.

Two events are conditionally independent event if:

P(A|C) x P(B|C) = P(AB|C)

Consider an example. Suppose A is the height of a child, and B is the number of words that the child knows. It seems that when A is high, B is high too (i.e. the two events are not independent). However, adding a single piece of information: the child's age (C) makes A and B completely independent. The height and the number of words known by the child are not independent, but they are conditionally independent if you provide the kid's age. In mathematical terms. P(A) and P(B) are not independent while P(A|C) and P(B|C) are independent.

3. Bayes' Rule

Bayes' formula is a rational method for updating or adjusting the probability of an event based on new information. According to Bayes' formula, the updated probability of an event given new information is:

$$P(\text{Event} \mid \text{Information}) = \frac{P(\text{Information} \mid \text{Event})}{P(\text{Information})} \times P(\text{Event})$$

Example

Consider a factory that has three assembly lines. The percentage of output produced at each assembly line is as follows: Line A = 45%, Line B = 35%, Line C = 20%. The output defective from each line is estimated to be 3%, 5%, and 4%, respectively. Given that the product is defective, what is the probability that it came from Line C?

Solution:

When dealing with questions related to Bayes' formula, the first step is to reproduce the information in probability notation:

P(Line A) = 0.45; P(Not Line A) = 0.55

P(Line B) = 0.35; P(Not Line B) = 0.65

P(Line C) = 0.20; P(Not Line C) = 0.80

P(Defective | Line A) = 0.03, P(Defective | Line B) = 0.05, P(Defective | Line C) = 0.04

P(Defective) = 0.45 x 0.03 + 0.35 x 0.05 + 0.20 x 0.04 = 0.039

Next write down the Bayes formula:

$$P(\text{Event} \mid \text{Information}) = \frac{P(\text{Information} \mid \text{Event})}{P(\text{Information})} \times P(\text{Event})$$

We then have to distinguish between the event and the information and plug the relevant values into the formula. In this case, the information is that the product is defective. Hence, the formula can be written as:

$$P(\text{Line C} \mid \text{Defective}) = \frac{[P(\text{Defective} \mid \text{Line C}) * P(\text{Line C})]}{P(\text{Defective})} = \frac{0.04 * 0.20}{0.039} = 20.51\%$$

Summary

LO: Describe an event and an event space.

A random variable is an uncertain quantity/number.

An outcome is the observed value of a random variable.

An event can be a single outcome or a set of outcomes.

An event space is the set of all possible outcomes and combinations of outcomes.

LO: Describe independent events and mutually exclusive events.

If the occurrence of one event does not influence the occurrence of the other event, then the two events are called independent events.

i.e. P(A|B) = P(A) or P(B|A) = P(B)

Multiplication rule for independent events: P(AB) = P(A) P(B)

Mutually exclusive events are events that cannot happen at the same time. If the two events are mutually exclusive, the joint probability: P(AB) is zero and the probability that either A or B will occur is simply the sum of the unconditional probabilities for each event:

P(A or B) = P(A) + P(B)

LO: Explain the difference between independent events and conditionally independent events.

If the occurrence of one event does not influence the occurrence of the other event, then the two events are called independent events.

Events that are not independent may be conditionally independent based on a third event. Two events are conditionally independent event if:

P(A|C) x P(B|C) = P(AB|C)

LO: Calculate the probability of an event for a discrete probability function.

The probability of an event = number of outcomes in that event/ total possible outcomes

For example, the probability of rolling a 2 is 1/6 = 16.7%. The probability of rolling an even number is 3/6 = 50%.

LO: Define, describe, and calculate a conditional probability.

The joint probability of two events A and B is expressed as:

P(AB) = P(A|B) P(B)

Rearranging the equation, we get the formula for computing conditional probabilities:

P(A|B) = P(AB) / P(B)

LO: Differentiate between conditional and unconditional probabilities.

Unconditional probability is the probability of an event occurring irrespective of the occurrence of other events. It is denoted as P(A). Unconditional probability is also called 'marginal' probability.

Conditional probability is the probability of an event occurring given that another event has occurred. It is denoted as P(A|B), which is the probability of event A given that event B has occurred.

LO: Explain and apply Bayes' rule.

Bayes' formula is a rational method for updating or adjusting the probability of an event based on new information. According to Bayes' formula, the updated probability of an event given new information is:

$$P(\text{Event} \mid \text{Information}) = \frac{P(\text{Information} \mid \text{Event})}{P(\text{Information})} \times P(\text{Event})$$

