Question 1 of 15

P1.T2.710.1

The following probability matrix contains the joint probabilities for random variables $X = \{2, 7, 12\}$ and $Y = \{1, 3, 5\}$:

		Y			
		1.0	3.0	5.0	
X	2.0	5.00%	15.00%	5.00%	25.0%
	7.0	10.00%	30.00%	10.00%	50.0%
	12.0	5.00%	15.00%	5.00%	25.0%
		20.0%	60.0%	20.0%	100.0%

We are informed that (X) and (Y) are independent. What is the expected value of the product of X and Y, $E(XY)$?

- **A.** 15.0
- **B.** 21.0
- **C.** 30.5
- **D.** 35.0

Explanation

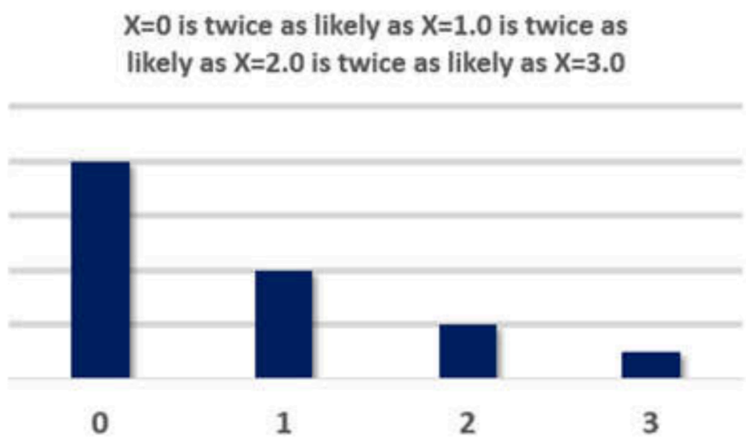
B is CORRECT.

Although we can sum the product of $f(x) \times X \times Y$ for all nine cells, it is much easier to seize on the property of independence: if X and Y are independent, then by definition $E(XY)$ = $E(X) \times E(Y) = 3.0 \times 7.0 = 21.0$.

Question 2 of 15

P1.T2.710.2

Peter is running the first draft of a Monte Carlo simulation and he wants a simple random variable to capture the loss frequency per week (i.e., the number of loss events) associated with an operational process. As he is just experimenting with his model, he does not have a probability density function in mind. Instead, he has a simple rule-based idea that he wants to express in a probability function. The number of losses per week will be either zero, one, two or three; $X = \{0, 1, 2, 3\}$. An outcome of $X = 3.0$ is the least likely; an outcome of $X = 2.0$ is twice as likely as $X = 3.0$; an outcome of $X = 1.0$ is twice as likely as $X = 2.0$, and finally an outcome of $X = 0$ is twice as likely as $X = 1.0$. This is illustrated below:



Assuming this qualifies as a probability distribution, which is **NEAREST** to the standard deviation of this random variable?

- **A.** Zero
- **B.** 0.625
- **C.** 0.750
- **D.** 0.930

Explanation

D is CORRECT.

Let a = constant such that $1a + 2a + 4a + 8a = 1.0$ per a probability distribution, and therefore $a = 1/15$.
The expected value of

$$E(X) = (1/15) \times 3.0 + (2/15) \times 2.0 + (4/15) \times 1.0 + (8/15) \times 0 = (3 + 4 + 4)/15 = 11/15 = 0.7333.$$

The expected value of

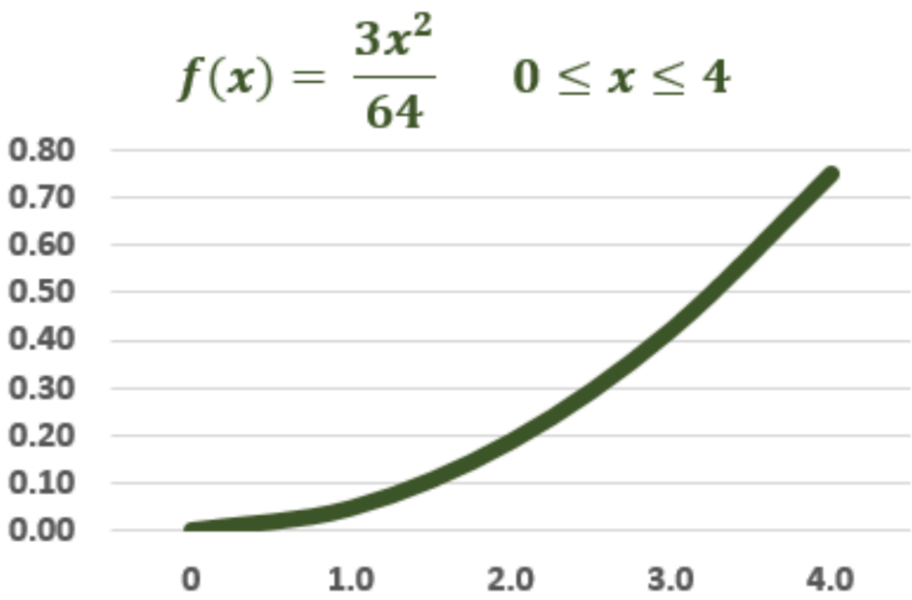
$$E(X^2) = (1/15) \times 3.0^2 + (2/15) \times 2.0^2 + (4/15) \times 1.0^2 + (8/15) \times 0^2 = (9 + 8 + 4)/15 = 21/15 = 1.40.$$

The variance of X, $\sigma^2(X) = E(X^2) - E(X)^2 = 1.40 - (11/15)^2 = 0.86222$ and the standard deviation equals $\sqrt{1.40 - (11/15)^2} = 0.92856$.

Question 3 of 15

P1.T2.710.3

Consider a random variable that represents the loss severity of a risky asset and is given by the continuous probability distribution $f(x) = 3x^2/64$ on the domain from zero to four:



What is this variable's mean and variance?

- A. $\mu = 1.0$ and $\sigma^2 = 3.50$
- B. $\mu = 2.0$ and $\sigma^2 = 2.75$
- C. $\mu = 3.0$ and $\sigma^2 = 0.60$
- D. $\mu = 3.6$ and $\sigma^2 = 0.25$

Explanation

C is CORRECT.

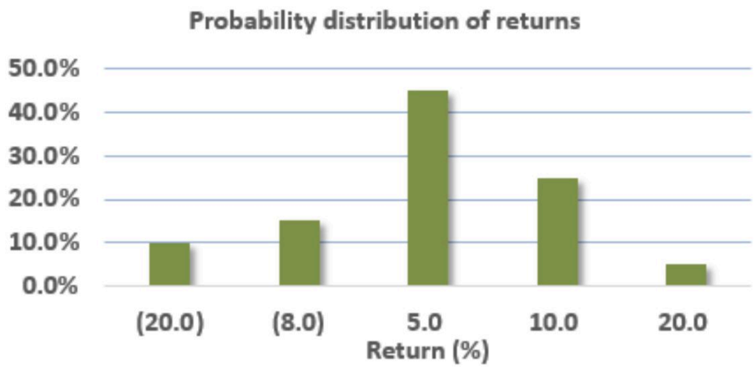
- To retrieve the mean μ, we integrate $xf(x)$

- To retrieve the variance, we integrate $(x - \mu)^2 f(x)$

Question 4 of 15

P1.T2.712.1

Consider the following discrete probability distribution of asset returns:



			Central moments ("about the mean")		
X	f(X)	X*f(X)	2nd	3rd	4th
			$(X - \mu_X)^2 \cdot f(X)$	$(X - \mu_X)^3 \cdot f(X)$	$(X - \mu_X)^4 \cdot f(X)$
(20.0)	10.0%	(2.00)	50.9	(1,146.7)	25,857.5
(8.0)	15.0%	(1.20)	16.7	(176.1)	1,858.2
5.0	45.0%	2.25	2.7	6.6	16.2
10.0	25.0%	2.50	13.9	103.4	770.1
20.0	5.0%	1.00	15.2	265.7	4,636.1
Sum:		2.55	99.3	(947.1)	33,138.2

As shown, this asset's expected return is +2.55%. Which is **NEAREST** to this variable's skew; aka, standardized third central moment?

- **A.** -2.25
- **B.** -0.96
- **C.** +0.33
- **D.** +1.06

Explanation

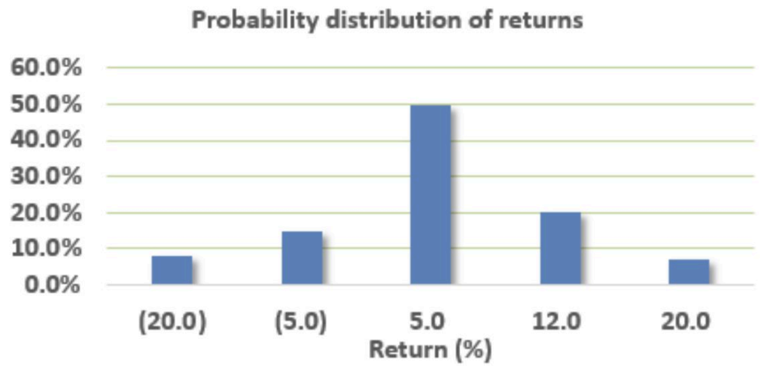
B is CORRECT.

Because $-947.1/99.3^{3/2} = -0.957$; i.e., negative or skewed to the left. We can eyeball this by noticing that the third central moment (displayed as -947.1) is divided by approximately 10^3 as the standard deviation is approximately $10 \approx \sqrt{99.3}$.

Question 5 of 15

P1.T2.712.2

Consider the following discrete probability distribution of asset returns:



			Central moments ("about the mean")		
X	f(X)	X*f(X)	2nd	3rd	4th
			$(X - \mu_X)^2 \cdot f(X)$	$(X - \mu_X)^3 \cdot f(X)$	$(X - \mu_X)^4 \cdot f(X)$
(20.0)	8.0%	(1.60)	45.9	(1,099.0)	26,321.6
(5.0)	15.0%	(0.75)	12.0	(107.5)	962.5
5.0	50.0%	2.50	0.6	0.6	0.6
12.0	20.0%	2.40	13.0	104.3	839.9
20.0	7.0%	1.40	18.0	289.4	4,645.1
Sum:		3.95	89.4	(812.2)	32,769.7

As shown, this asset's expected return is +3.95%. Which is **NEAREST** to this variable's kurtosis; aka, standardized fourth central moment?

- **A.** -1.47
- **B.** +2.81
- **C.** +4.10
- **D.** +7.50

Explanation

C is CORRECT.

Because $32{,}769.7/89.4^2 = 4.1001$; i.e., heavy-tailed with "excess kurtosis" of $4.10 - 3.0 = 1.10$.

Question 6 of 15

P1.T2.712.3

Portfolio manager Peter manages a large portfolio with 100 component positions. He is interested in analyzing the non-trivial cross moments in the portfolio (trivial cross-moments are the position's coskew/cokurtosis with itself, which is simply the position's standard skew or kurtosis, so these are analogous to the diagonal of a covariance matrix which is mere variances. Each of the following statements is true **EXCEPT** which is inaccurate?

- **A.** Between any two ($n = 2$) positions in the portfolio, the number of non-trivial coskew moments between them is two
- **B.** Between any two ($n = 2$) positions in the portfolio, the number of non-trivial cokurtosis moments between them is three
- **C.** Given a sub-portfolio consisting of any two positions, lower coskew values (i.e., where positives are gains and negatives are losses) imply greater risk for the sub-portfolio
- **D.** Although it is easy to estimate this portfolio's set of higher-order cross moments, standard skew and kurtosis are preferred because they are BLUE and the informational utility of coskew and cokurtosis is negligible

Explanation

D is CORRECT.

This is actually not easy because for 100 positions there exist fully 171,600 coskew cross moments and 4,421,175 cokurtosis cross moments!

Also, according to Miller, three is indeed often information utility in coskew and cokurtosis (alas there is a formidable curse of dimensionality in accessing them). The reference to BLUE is a red herring; BLUE refers to estimator properties.

In regard to (A), (B) and (C), each is TRUE.

In regard to true (A) and (B), the number of non-trivial cross moments is given by $k = \frac{(m+n-1)!}{m!(n-1)!} - n$. But in the case of only two variables, say (X) and (Y), the total number of coskew cross moments is four, which includes two trivial: $S(XXX), S(XXY), S(XYY), S(YYY)$. The total number of cokurtosis cross moments is five, which includes two trivial: $K(XXXX), K(XXXY), K(XXYY), K(XYYY), K(YYYY)$.

Question 7 of 15

P1.T2.303.1

Assume a continuous probability density function (pdf) is given by $f(x) = ax$ such that $0 \leq x \leq 12$, where (a) is a constant (we can retrieve this constant, knowing this is a probability density function):

$$f(x) = ax \quad \text{s.t.} \quad 0 \leq x \leq 12$$

What is the mean of (x)?

- **A.** 9.3
- **B.** 5.5
- **C.** 6.0
- **D.** 8.0

Explanation

D is CORRECT.

If this is a valid probability (pdf) then $a \times (1/2) \times x^2$ evaluated over [0,12] must equal one: $a \times (1/2) \times 12^2 = 1.0$, and $a = 1/72$.

Therefore, the pdf function is given by $f(x) = x/72$ over the domain of [0,12].
The mean = $\int x f(x)\,dx = x \times (1/72) \times x = \int x^2/72\,dx = x^3/216\big|_0^{12} = 12^3/216 = 8.0$

Question 8 of 15

P1.T2.303.2

Assume a continuous probability density function (pdf) be given by $f(x) = ax^2$ such that $0 \le x \le 3$, where (a) is a constant (that we can find).

$$f(x) = ax^2 \quad \text{s.t.} \quad 0 \le x \le 3$$

Let us arbitrarily define the unexpected loss (UL) as the difference between this distribution's mean and its 5.0% quantile function, i.e., $UL(X) = \text{mean}(X) - \text{inverse CDF}_{5\%}(X)$. We could call this a 95% relative VaR since it is relative to the mean. What is this UL?

- A. 0.62
- B. 1.14
- C. 2.05
- D. 3.37

Explanation

B is CORRECT.

If this is a valid pdf then $a^2 \times (1/3) \times x^3$ evaluated over [0,3] must equal one:
$a \times (1/3) \times x^3 = 1.0$, and $a = 3/27 = 1/9$.

Therefore, the pdf function is given by $f(x) = x^2/9$ over the domain of [0,3]
The mean (mu) = $\int x f(x)\,dx = x \times x^2/9 = \int x^3/9\,dx = (1/9)(1/4)x^4\Big|_0^3 = 3^4/36 = 9/4.$

For 5% quantile, we need value of (m) such that the integral of $f(x)\,dx$ over [0,m] = 0.05.
So, we need (m) so that $x^2/9\,dx$ over [0,m] = 0.05, and
$x^3/27$ over [0,m] = 0.05, and
$m^3/27 = 0.05$, therefore $m = (27 \times 0.05)^{1/3} = 1.11$
$UL(5\%) = \text{mean} - 1.11 = 9/4 - 1.11 = 1.14$

Question 9 of 15

P1.T2.20.3.1

Assume a continuous random variable over the domain $\{6 < X < 10\}$ is given by $f(x) = ax$ where (a) is a constant, i.e., $\{f(x) = ax \mid x \in (6, 10)\}$. What is the $\Pr(X \leq 8)$?
Bonus: what is the variable's expected value?

- **A.** 19.53%
- **B.** 43.75%
- **C.** 50.00%
- **D.** 56.25%

Explanation

B is CORRECT.

The (indefinite) integral evaluated over (6,10) must equal one such that $a(10^2/2 - 6^2/2) = 1.0$ --> $a = 1.0/(10^2/2 - 6^2/2) = 1/(100/2 - 36/2) = 1/32$. Because $a = 1/32$, the density function is given by $f(x) = x/32$ over [6,10]

The cumulative distribution function is given by $F(x) = x^2/64 - c$, but $F(6)$ must be zero and $F(10)$ must be 1.0; therefore, the CDF is $F(x) = x^2/64 - 0.5625$. And $F(8) = 8^2/64 - 0.5625 = 1.0 - 0.5625 = 0.43750$ or 43.75%

In regard to the bonus question, we need to integrate $xf(x)$ or $x \times x/32 = x^2/32$ which is $x^3/32 \times 1/3 = x^3/96$, and evaluate this over [6,0] such that the mean is $10^3/96 - 6^3/96 = 10.41667 - 2.25 = 8.1667$. See Absolutely continuous case at; https://en.wikipedia.org/wiki/Expected_value.

By the way, what is the median? Given that the CDF $F(X) = x^2/64 - 0.5625$, we can solve for $x = \sqrt{[F(X) + 0.5625] \times 64}$ which solve for any quantile (i.e., x) as a function of the CDF. The median is the 50th percentile such that the associated quantile (aka, the median) is given by, $x = \sqrt{[0.50 + 0.5625] \times 64} = 8.24621$. So, this distribution has a mean of 8.1667 and a median of 8.2462.

Question 10 of 15

P1.T2.20.3.2

A discrete random variable is characterized by the probability mass function (pmf) as given by $f(x) = xa$, and its domain is the set of integers {6, 7, 8, 9, and 10}. What is the variable's expected value?

- **A.** 6.67
- **B.** 8.00
- **C.** 8.25
- **D.** 9.33

Explanation

C is CORRECT.

We know that $6a + 7a + 8a + 9a + 10a = 1.0$ such that $a = 1/40 = 0.0250$ and the pmf is given by $f(x) = x/40 = 0.0250x$.
The expected value equals $(1/40)(6^2 + 7^2 + 8^2 + 9^2 + 10^2) = 8.25$

Question 11 of 15

P1.T2.20.4.3

Let Z be a random variable that is a linear function of random variables X and Y, where $Z = 3X + 7Y$? If the standard deviation of X and Y, respectively, are $\sigma(X) = 4.0$ and $\sigma(Y) = 5.0$ and the correlation between X and Y is $\rho(X, Y) = 0.50$, then what is the standard deviation of Z, $\sigma(Z)$?

- **A.** 6.40
- **B.** 7.81
- **C.** 37.00
- **D.** 42.30

Explanation

D is CORRECT.

$\sigma(Z) = \sqrt{\sigma^2(Z)}$,

and

$$\sigma^2(Z) = \sigma^2(3X + 7Y) = 3^2\sigma^2(X) + 7^2\sigma^2(Y) + 2 \times 3 \times 7 \times \text{cov}(X, Y) = 9 \times 4^2 + 49 \times 5^2 + 2 \times 3 \times 7 \times (4 \times 5 \times 0.5) = 1{,}789.0$$

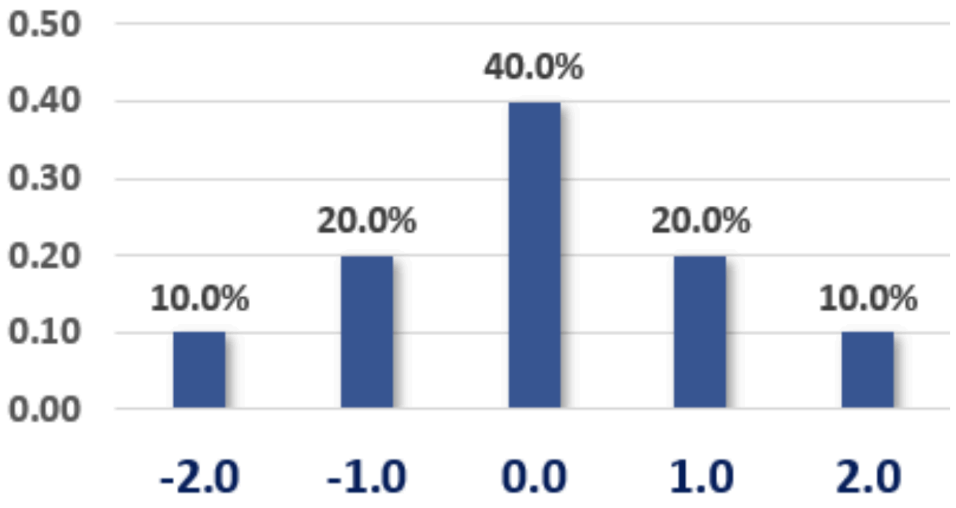
and $\sigma(Z)$ = $\sqrt{1{,}789.0} = 42.30$.

This applies the basic variance properties that we need to know; see https://en.wikipedia.org/wiki/Variance#Properties
In this case, $\sigma^2(aX + bY) = a^2\sigma^2(X) + b^2\sigma^2(Y) + 2ab\,\text{Cov}(X, Y)$, where (a) and (b) are constants.

Question 12 of 15

P1.T2.20.4.1

Consider the probability mass function (pmf) below. For example, $\Pr(X = -1.0) = 20.0\%$.



As we can see, this distribution is symmetrical, so we know that its skewness is zero without performing any calculations. We are told the variance is 1.20 (although we can calculate the variance). What is this distribution's kurtosis?

- **A.** Zero
- **B.** 2.50
- **C.** 3.60
- **D.** 4.40

Explanation

B is CORRECT.

$(-2)^4 \times 10.0\% + (-1)^4 \times 20.0\% + 0^4 \times 40.0\% + 1^4 \times 20.0\% + 2^4 \times 10.0\% = 3.60$, but we need to standardize such that $\text{kurtosis} = 3.60/1.20^2 = 2.50$.

Kurtosis is the fourth central moment (aka, fourth moment about the mean) standardized by the standard deviation raised to the fourth power (aka, square of the second moment). So, in the numerator we have $E[X - \mu(x)]^4$, which is the fourth central moment; and this is divided by $\sigma(X)^4$ in order to standardize the moment. See https://en.wikipedia.org/wiki/Standardized_moment

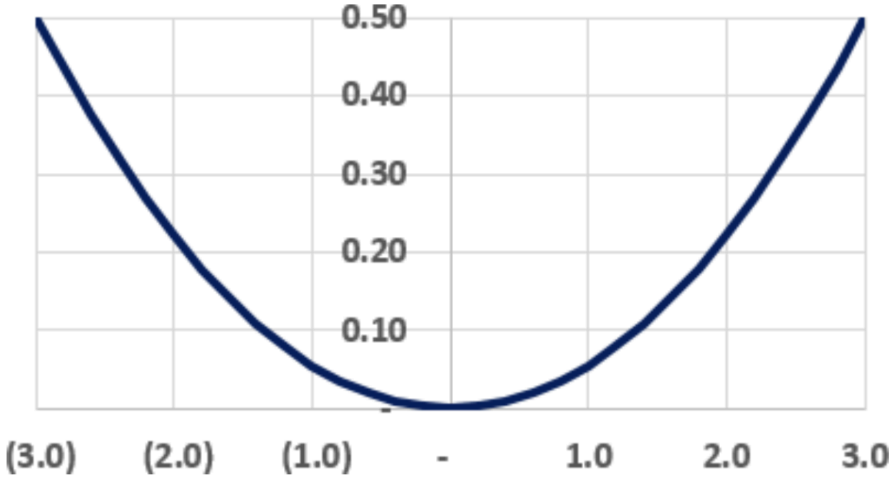
Question 13 of 15

P1.T2.20.4.2

Peter decides that the payoff of his option straddle strategy can be approximated by the probability density function (pdf) illustrated below. The function is: $f(x) = ax^2$ over the real domain $\{-3 < X < +3\}$; i.e., $\{f(x) = ax^2 \mid x \in (-3.0, 3.0)\}$. He used x-squared because he wants the shape of a parabola, but how does he determine the value of the constant, a? It cannot be anything! Because this is a probability distribution, it must be true that the integral, $ax^3/3$, when evaluated over $[3, -3]$ equals one: the area under the pdf curve must equal one, and the integral of the pdf is the cumulative distribution function (CDF).

Given that $a3^3/3 - a(-3)^3/3 = 1.0$, we must have $9a - (-9a) = 1.0$ such that $a = 1/18$. Therefore, a is $1/18$ and his probability density function is $f(x) = x^2/18$. But is $x^3/54$ the integral? He notices that if $F(x) = x^3/54$ then $F(3) = 27/54 = 0.5$ and $F(-3) = -0.5$, but the CDF requires that $F(3) = 1.0$ and $F(-3.0) = 0$. Seeing this, he realizes that the indefinite integral includes a constant and is given by $ax^3 + c$, or specifically, $x^3/54 + c$. Now it is possible for $3^3/54 + c = 1.0$, if $c = 1.0 - 0.5 = 0.5$. Finally, he can specify the exact pdf, $f(x) = x^2/18$, as the derivative of the correct CDF, $F(x) = x^3/54 + 0.5$, so that $F(3) = 1.0$ and $F(-3) = 0$.

Value-at-risk (VaR) is the distributional quantile associated with a specified probability. The inverse CDF (aka, quantile function) returns the quantile implied by a probability, $q = F^{-1}(p)$, whereas the CDF returns the probability associated with a quantile: $p = F(q)$, or in this case, $p = F(X)$. In this distribution, losses are on the left (negative payoffs), so the 95.0% VaR is the quantile implied by a 5.0% probability: $X = F^{-1}(p) = F^{-1}(0.050)$. For this distribution, what is the 95.0% VaR, in this case, the 0.050 quantile?



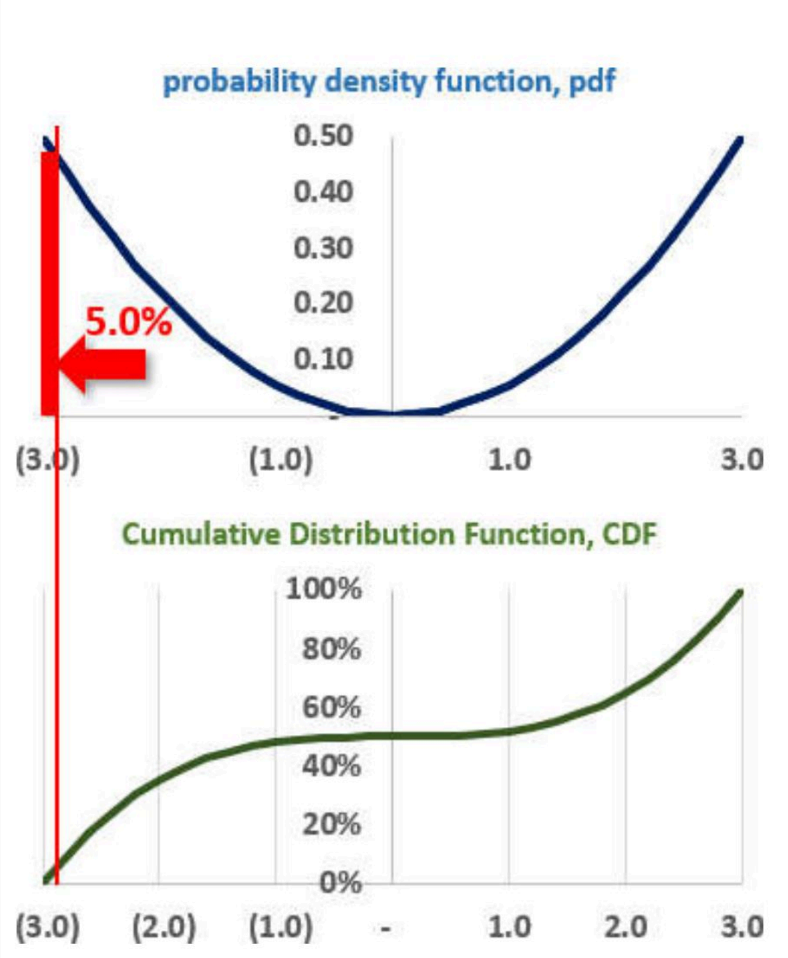
- [x] **A.** -2.90
- [ ] **B.** -2.70
- [ ] **C.** -1.50
- [ ] **D.** +0.33

Explanation

A is CORRECT.

$p = 0.050 = x^3/54 + 0.5$, such that $x = [(0.050 - 0.50) imes 54]^{1/3} = -2.8965$.

In other words, if we want the 95.0% VaR, we want the quantile (i.e., the value on the X-axis. In this case, it starts at -3.0) such that on the pdf, the area to the left (under the curve) is 5.0% which corresponds to 5.0% on the Y-axis of the CDF. See below. But that is simply the CDF! The CDF $F(x)$ is the probability: at 5.0% --> $5.0\% = x^3/54 + 0.5$, and we can solve for the implied quantile. We return with $x = -2.8965$ which signifies, per $X = F^{-1}(p)$, that $q = -2.8965 = F^{-1}(5.0\%)$. The probability distribution is a mathematical description of a random variable (e.g., the CDF must start at zero and end at 1.0). In this question, we've identified -2.8965 as the quantile we expect to "do better than" 95.0% of the time. Or, put another way, we expect to do worse than -2.8965 only 5.0% of the time.


probability density function, pdf
0.50
0.40
0.30
0.20
0.10
5.0%
(3.0)
(1.0)
1.0
3.0
Cumulative Distribution Function, CDF
100%
80%
60%
40%
20%
0%
(3.0)
(2.0)
(1.0)
-
1.0
2.0
3.0

Question 14 of 15

P1.T2.20.3.3

Ralph is an analyst who wants to characterize a random variable with a discrete probability distribution where the only outcomes are {0, 1, 2, 3, and 4}. The Poisson distribution with a mean of 1.0 is a surprisingly good fit as it gives $\Pr(X=0)=\Pr(X=1)=36.8\%$. The problem with the Poisson is that it has a long tail such that the $\Pr(X\geq 5)=0.366\%$; for example, there is a very tiny possibility (1.0E-08) that the outcome could exceed ten given the Poisson's support is the entire set of natural numbers. To enforce a true probability distribution that is bounded at four, Ralph simply rounds the pmf densities and, due to sheer luck, when rounded (to two digits) the first five outcomes (including zero) sum exactly to 100.0%. The ensuing probability distribution is the following, which might be dubbed a "truncated Poisson" with $\lambda=1.0$:

X	f(X)
0	37.0%
1	37.0%
2	18.0%
3	6.0%
4	2.0%
	100.0%

The exact mean (aka, expected value or weighted average) of this variable is 0.990, but let's assume its average is a round 1.0. What is the skewness of this variable?

- A. -0.774
- B. Zero
- C. +0.869
- D. +2.440

Explanation

C is CORRECT.

C. True: $\text{skewness} = +0.869$

The third central moment

$$= 37.0\% \times (0-1)^3 + 37.0\% \times (1-1)^3 + 18.0\% \times (2-1)^3 + 6.0\% \times (3-1)^3 + 2.0\% \times (4-1)^3 = 0.830.$$

Skewness standardizes the third central moment by dividing by the standard deviation cubed, σ^3, or equivalently, variance raised to the 2/3-rd power because $\sigma^3 = (\sigma^2)^{3/2}$.

In this case, the variance (aka, second central moment

$$= 37.0\% \times (0-1)^2 + 37.0\% \times (1-1)^2 + 18.0\% \times (2-1)^2 + 6.0\% \times (3-1)^2 + 2.0\% \times (4-1)^2 = 0.970.$$

Therefore, the skewness = $0.830 \div 0.970^{1.5} = 0.8688$

Question 15 of 15

P1.T2.303.3

Assume the following probability density function (pdf) for a random variable X:

$$f(x) = \frac{x}{18} \quad s.t. \quad 0 \leq \text{x} \leq 6$$

What is the variance of X?

- **A.** 2.0
- **B.** 3.3
- **C.** 4.1
- **D.** 5.7

Explanation

A is CORRECT.

A. 2.0
Mean (mu) = $\int x f(x)\, dx = x \times x/18 = x^2/18$ evaluated from [0,6] = $(1/54)x^3$ from [0,6] = $6^3/54 = 4.0$

Variance = $\int (x-\mu)^2 f(x)\, dx$ evaluated from [0,6] = $(1/18)[x^4/4 - 8x^3/3 + 8x^2]$ from [0,6] = $(1/18)[6^4/4 - 8 \times 6^3/3 + 8 \times 6^2] = 2.0$