Quantitative Methods

Learning Module 1
Returns of Financial Assets and Instruments

Introduction

This learning module explains the meaning of and calculations for various measures of return: the goal of investing. While the discussion of quantitative methods becomes more complex throughout the reading, it begins with this most defining measure.

Financial Returns

LOS: Describe, compare, and interpret returns.

Financial returns measure the change in value of an investment over time and represent the compensation investors receive for deferring consumption and bearing risk. In practice, returns arise from two sources: changes in the price of an asset, and cash flows distributed during the holding period. Although the mechanics of calculating returns are straightforward, their interpretation depends on the time horizon, compounding assumptions, and the method of aggregation across periods.

A clear understanding of return measurement is essential since it underlies performance evaluation, portfolio construction, and risk assessment.

Types of Returns for Financial Assets, Instruments, and Indicators

Financial assets represent claims to future cash flows. Equity securities provide ownership claims on a company's residual profits, whereas debt securities provide contractual claims to interest and principal repayment. Hybrid securities combine features of both debt and equity. These assets generate returns either through price appreciation, cash distributions, or both.

Financial instruments are the standardized, tradable contracts through which financial assets are exchanged. For example, shares of common stock and corporate bonds are instruments that represent underlying claims. Standardization enhances liquidity and market efficiency.

Financial indicators, such as interest rates, exchange rates, and market indexes, differ from financial instruments in that they do not themselves generate cash flows. Rather, they measure or influence the economic environment in which assets are priced.

Understanding these distinctions clarifies what generates returns and what merely affects them.

Total Return

The total return earned over a specific time interval incorporates both price appreciation and income received during the period.

Total return can be decomposed into two components:.

- Capital appreciation return = $(P_1 - P_0)/P_0$
- Capital distribution return = Inc/P_0

For equities, income return is often called the dividend yield. For bonds, it is commonly referred to as the current yield.

Some investments generate return solely from price changes, such as non-dividend-paying stocks. Others generate return primarily through income, such as certain fixed-income instruments held to maturity. Most financial assets involve both components.

Example 1 Decomposing total return

An investor purchases a stock for $50. Over the next year, the stock price rises to $56, and the stock pays a dividend of $2.

1. Calculate the total return.
2. Decompose the return into capital appreciation return and capital distribution return.

Solution

The total return is calculated as:

$$\text{HPR} = \frac{P_1 - P_0 + \text{Inc}}{P_0}$$

$$= \frac{56 - 50 + 2}{50} = \frac{8}{50} = 0.16 = 16\%$$

The capital appreciation return equals:

$$\frac{56 - 50}{50} = 12\%$$

The capital distribution return equals:

$$\frac{2}{50} = 4\%$$

Thus, the total return of 16% consists of 12% capital appreciation and 4% dividend yield.

Total return reflects both price movement and income received. Ignoring the dividend would understate performance.

Expected returns, also referred to as *ex ante* returns, represent forecasts of future performance. These expectations reflect required compensation for bearing risk and are influenced by economic conditions, company fundamentals, and investor sentiment.

Actual returns, or *ex post* returns, are the realized outcomes at the end of the holding period. Since financial markets are uncertain, actual returns often differ from expected returns. The variability between expected and actual outcomes constitutes investment risk.

Investment decisions are made based on expected returns, but performance evaluation relies on actual returns.

An unrealized return represents a gain or loss that has occurred in market value but has not yet been locked in through sale. These are sometimes referred to as paper gains or losses. When an asset is sold and the gain or loss is crystallized, the return becomes realized. Capital distributions such as dividends and coupon payments are realized when received.

Arithmetic and Geometric Holding Period Returns

When returns are observed over multiple periods, analysts often calculate an average return. The arithmetic mean return is the simple average of periodic returns:

Arithmetic mean

$$\overline{X} = \frac{x_1 + x_2 + \ldots + x_n}{n}$$

This measure answers the question: "What was the average return per period?" It is particularly useful when estimating expected returns for a single future period, as it is an unbiased estimateor of the expected value under certain statistical assumptions. However, the arithmetic mean does not reflect the effect of compounding across periods.

The geometric mean return measures the compound rate of growth over multiple periods. It is calculated as:

$$G = \sqrt[n]{[(1 + x_1)(1 + x_2)\ldots(1 + x_n)]} - 1$$

This measure reflects the constant rate of return that would produce the same ending wealth over the observed time horizon.

Example 2 Arithmetic vs. geometric mean returns

An investment earns the following annual returns over three years:

Year 1: +20%

Year 2: −10%

Year 3: +15%

1. Calculate the arithmetic mean return.
2. Calculate the geometric mean return.
3. Interpret the difference.

Solution

The arithmetic mean return equals:

$$\frac{0.20 - 0.10 + 0.15}{3} = \frac{0.25}{3} = 8.33\%$$

To calculate the geometric mean:

$$(1.20)(0.90)(1.15) = 1.242$$
$$\bar{r}_G = (1.242)^{1/3} - 1$$
$$= 1.0749 - 1 = 7.49\%$$

The arithmetic mean (8.33%) exceeds the geometric mean (7.49%). The difference arises since volatility reduces compound growth. The geometric mean more accurately reflects the true compound growth rate over the three-year period.

Unlike the arithmetic mean, the geometric mean fully accounts for compounding. Therefore, it is more appropriate for long-term performance evaluation.

A fundamental relationship is that the geometric mean return is always less than or equal to the arithmetic mean return. The difference between the two is created by compounding, and it increases as volatility increases. This occurs due to negative returns reducing the capital base upon which future returns are earned.

Compounding: Daily, Monthly, Quarterly, and Annual Returns

To facilitate comparisons across investments measured over different time intervals, returns are often annualized. If a periodic return r_{period} is observed and there are *c* such periods in a year, the annualized return is:

$$r_{\text{annualized}} = (1 + r_{\text{period}})^c - 1$$

Annualization assumes that returns are reinvested at the same rate and that the periodic return persists throughout the year. While this assumption allows for comparability, it may not be realistic in volatile markets.

When converting from an annual return to a periodic return, the inverse operation is used:

$$r_{\text{period}} = (1 + r_{\text{annual}})^{1/c} - 1$$

Example 3 Annualizing a periodic return

An investor earns a 2% return over one month.

1. Calculate the annualized return.
2. Explain the assumption embedded in annualization

Solution

Monthly compounding implies 12 periods per year:

$$r_{\text{annualized}} = (1.02)^{12} - 1$$

$$= 1.2682 - 1 = 26.82\%$$

The annualized return equals 26.82%.

This calculation assumes the 2% monthly return can be earned and reinvested at the same rate every month for a full year. In practice, monthly returns fluctuate, so annualization may overstate realistic expectations.

Continuous Compounding

As compounding frequency increases, the effective annual return approaches a theoretical limit known as continuous compounding. The continuously compounded return between two prices is defined as:

$$\bar{r} = \ln\left(\frac{P_1}{P_0}\right)$$

This is equivalent to $\ln(1 + r)$, where r is the simple holding period return.

Continuously compounded returns possess a useful property: they are additive over time. The continuously compounded return over multiple periods equals the sum of the individual period log returns. This additive property simplifies mathematical modeling and statistical analysis.

To convert from a continuously compounded return back to a simple return, use:

$$r = e^{r} - 1$$

Although log returns are widely used in financial theory and quantitative modeling, simple returns are more intuitive for performance reporting.

Example 4 Continuously compounded returns

A stock price increases from $100 to $108 over one year.

1. Calculate the simple return.
2. Calculate the continuously compounded return.
3. Verify the relationship between the two.

Solution

The simple return is:

$$\frac{108 - 100}{100} = 8\%$$

The continuously compounded return is:

$$\ln\left(\frac{108}{100}\right) = \ln(1.08) = 0.07696 = 7.696\%$$

To convert back:

$$\mathrm{e}^{0.07696} - 1 = 0.08 = 8\%$$

The continuously compounded return is slightly lower than the simple return but produces the same ending value when converted back. Log returns are additive across time, which makes them particularly useful in quantitative modeling.

Summary

Financial returns consist of price changes and income. The total return measures return from capital appreciation and capital distribution over a single interval. Arithmetic averages estimate expected one-period returns, whereas geometric averages measure compound growth over time. Annualization allows comparison across time intervals but relies on reinvestment assumptions. Continuously compounded returns simplify multi-period analysis through additivity.

A thorough understanding of these return measures is essential for interpreting investment performance and forms the foundation for later study of risk, portfolio theory, and asset pricing.

Common Return Measures

LOS: Describe, compare, and interpret required rates of return, risk-free rates, risk premiums, and inflation.

This section examines commonly used return measures and shows how they are adjusted for economic and practical realities such as inflation, taxes, fees, and leverage. While earlier sections focus on calculating returns, this section addresses how returns should be interpreted and adjusted to reflect what investors actually earn and require.

The discussion begins with the concept of the required rate of return, which establishes the benchmark against which investments are evaluated. It then considers excess returns, real returns, gross versus net returns, after-tax returns, and leveraged returns. Together, these concepts provide a more complete framework for evaluating investment performance.

The Required Rate of Return

Investors face a fundamental choice: consume today or defer consumption in exchange for future purchasing power. Deferring consumption involves opportunity cost. To justify postponing consumption, an investor must earn at least the required rate of return—the minimum return necessary to accept an investment.

Issuers of capital, such as corporations and governments, must offer returns that meet investors' required rates. Otherwise, investors will allocate their funds elsewhere. The required rate of return therefore serves as the equilibrium compensation for time and risk.

The required rate of return can be decomposed into two components: the risk-free rate and one or more risk premiums. The risk-free rate represents compensation for deferring consumption with no uncertainty regarding default or reinvestment. In practice, short-term government securities—such as Treasury bills—are often used as proxies since they carry minimal default risk and limited interest-rate sensitivity.

The nominal risk-free rate itself can be separated into the real risk-free rate and expected inflation. The real risk-free rate reflects time preference: the compensation required for deferring real (inflation-adjusted) consumption. The relationship is:

$$r_{rf} = \frac{1 + r_f}{1 + \text{Inflation}} - 1$$

where r_f is the nominal risk-free rate.

A common approximation is:

$$r_{rf} \approx r_f - \text{Inflation}$$

This approximation is convenient when inflation and interest rates are small, but it ignores compounding effects and becomes less accurate when rates are large or horizons are long.

In addition to the real risk-free rate, investors require compensation for bearing risk. These risk premiums vary by asset class and economic environment. For equities, examples include the market risk premium, size risk premium, and value risk premium. For bonds, examples include the default risk premium, liquidity premium, and maturity risk premium.

In simplified form, the required rate of return can be expressed as:

$$r_i \approx r_{rf} + \text{Inflation} + \sum f_{ij} r_{pj}$$

where each term reflects exposure to a specific risk factor and its associated premium.

Since economic conditions evolve, both risk premiums and required returns change over time.

Example 5 Calculating the real risk-free rate

Suppose the nominal risk-free rate is 4%, and expected inflation is 1.5%.

The precise real risk-free rate is:

$$\frac{1.04}{1.015} - 1 = 2.46\%$$

The approximation gives:

$$4\% - 1.5\% = 2.5\%$$

The difference appears small (0.04%), but over long horizons this difference compounds and can meaningfully affect projected wealth.

Excess Return

Excess return measures the performance of an investment relative to a benchmark. Often, the benchmark is the risk-free rate, but it may also be a market index or another reference portfolio.

The precise calculation of excess return is:

$$\frac{1 + \mathrm{r}_i}{1 + \mathrm{r}_f} - 1$$

An approximation is simply:

$$\mathrm{r}_i - \mathrm{r}_f$$

The approximation works reasonably well when returns are small, but the precise version accounts for compounding.

Excess returns are central in performance evaluation. A positive excess return indicates compensation above the benchmark; a negative excess return suggests underperformance.

Example 6 Calculating excess return

Assume a stock earns 12%, and the risk-free rate is 3%.

The approximate excess return is:

$$12\% - 3\% = 9\%$$

The precise calculation is:

$$\frac{1.\ 12}{1.\ 03} - 1 = 8.\ 74\%$$

The difference reflects compounding effects. For short horizons and moderate returns, the approximation is often sufficient.

Real Returns

Nominal returns measure percentage change in monetary terms, but inflation erodes purchasing power. Real returns adjust for inflation and measure the change in real wealth.

The real return is calculated as:

$$r_{\text{real}} = \frac{1 + r_i}{1 + \text{Inflation}} - 1$$

This measure allows for comparison of returns across periods with different inflation rates.

Inflation varies over time, so evaluating nominal returns alone can be misleading. A 10% nominal return during 8% inflation produces far less real wealth growth than the same nominal return during 2% inflation.

Example 7 Nominal vs. real return

An investment earns 9% during a year in which inflation is 6%.

The real return is:

$$\frac{1.09}{1.06} - 1 = 2.83\%$$

Although the nominal return appears strong, real purchasing power increased by less than 3%.

Gross and Net Return

Investors rarely earn the full return generated by underlying assets. Investment managers charge fees, and trading generates transaction costs. These costs reduce investor returns.

Gross return is the return earned after transaction costs but before management and administrative fees. It is often used to evaluate manager skill since it excludes operational expenses.

Net return reflects the return actually earned by the investor after all fees and expenses.

In simplified terms:

Total return − Transaction costs = Gross return
Gross return − Management fees − Administrative expenses = Net return

Costs compound over time. Even modest annual fees can significantly reduce long-term wealth accumulation.

Example 8 Gross vs. net return

A fund earns a total return of 11%. Transaction costs are 0.40%, management fees are 1.00%, and administrative expenses are 0.60%.

Gross return:

$$11\% - 0.40\% = 10.60\%$$

Net return:

$$10.60\% - 1.00\% - 0.60\% = 9.00\%$$

Although the portfolio generated 11%, the investor earned only 9%.

Pretax and After-Tax Nominal Return

Taxes further reduce investor returns. Capital gains and income distributions (such as dividends and interest) may be taxed at different rates. Tax systems vary across jurisdictions and investors, so after-tax returns differ from one investor to another.

The after-tax nominal return can be expressed as:

$$r_{\text{after tax}} = r_{\text{price}}(1 - \text{Tax}_{\text{cg}}) + r_{\text{capital}}(1 - \text{Tax}_{\text{income}})$$

Capital gains taxes apply to realized price appreciation, while income taxes apply to distributions.

Since taxes apply only to realized gains, deferring realization can enhance after-tax compounding. Tax-efficient portfolio management therefore seeks to minimize unnecessary turnover and exploit favorable tax treatment.

Example 9 After-tax return

An investor buys a stock at \$100 and sells it at \$120 after receiving a \$4 dividend. Capital gains are taxed at 25% and dividends at 20%.

Price return:

$$(120 - 100)/100 = 20\%$$

Income return:

$$4/100 = 4\%$$

After-tax return:

$$20\%(1 - 0.25) + 4\%(1 - 0.20)$$

$$= 15\% + 3.2\% = 18.2\%$$

The total nominal return before tax was 24%, but after tax the investor retains 18.2%.

Leveraged Returns

Leverage involves using borrowed funds to increase the size of an investment. Leverage magnifies gains and losses since returns are earned on a larger asset base, while borrowing costs must be paid regardless of performance.

If the portfolio return r_p exceeds the borrowing cost r_d, leverage increases equity returns. If the portfolio return is below the borrowing cost, leverage amplifies losses.

The leveraged return is:

$$r_{LP} = r_p + \frac{V_D}{V_E}(r_p - r_d)$$

where V_E is equity and V_D is debt.

Leverage increases both expected return and risk. It does not create value by itself. Value arises only if the return on invested capital exceeds the cost of borrowing.

Example 10 Leveraged return

An investor invests $70,000 of personal capital and borrows $30,000 at 6% to create a $100,000 portfolio. The portfolio earns 10%.

Unleveraged return: 10%.

Leveraged return:

$$0.10 + \frac{30,000}{70,000}(0.10 - 0.06) = 0.117143 \approx 11.71\%$$

Leverage increases the return as the portfolio return exceeds borrowing cost.

If the portfolio earned 4% instead, leverage would produce a lower return than the unleveraged portfolio, demonstrating its risk-amplifying nature.

Summary

The required rate of return represents the minimum compensation investors demand for deferring consumption and bearing risk. It consists of a real risk-free rate, expected inflation, and risk premiums. Excess returns measure performance relative to a benchmark. Real returns adjust for inflation and reflect purchasing power. Gross and net returns incorporate the effects of fees and expenses. After-tax returns reflect the impact of taxation on price and income components. Leveraged returns magnify both gains and losses by incorporating borrowed capital.

Together, these measures provide a comprehensive framework for evaluating investments in realistic economic conditions.

Learning Module 2
Types of Financial Returns

Types of Returns for Financial Assets, Instruments, and Indicators

LOS: Calculate, compare, and interpret different types of returns for financial assets, instruments, and indicators.

Investors earn returns from two primary sources: **price appreciation** and **capital distribution**.

- Price returns arise from changes in the market value of a financial asset.
- Capital distribution returns arise from cash flows paid to investors, such as dividends or coupon payments.

The relative importance of these two components varies across asset classes and across time.

Price return reflects changes in an asset's price that are independent of any income distributions. For equities, price changes are influenced by firm performance, economic conditions, investor expectations, and market sentiment. For bonds, price changes are primarily driven by movements in interest rates and changes in the issuer's credit quality. Since price changes are uncertain and variable, they are often the central focus of risk analysis.

Capital distribution returns may be either fixed or variable:

- Fixed capital distributions are most common in debt instruments and consist of predetermined interest or coupon payments. Although the contractual rate is fixed, realized returns may differ if the issuer defaults.
- Variable capital distributions are most common in equities, where dividends may increase, decrease, or be suspended entirely. Some debt instruments, such as floating-rate bonds, also provide variable capital distributions.

Types of Returns: Financial Assets

Financial assets are commonly categorized into cash, equities, debt (bonds), and hybrid instruments. Each asset type generates returns in distinct ways.

Cash consists of currency and non-interest-bearing deposits. Cash itself does not generate return unless invested. Short-term investments, such as Treasury bills or interest-bearing accounts, generate return primarily through capital distributions rather than price appreciation.

Equities represent ownership interests in firms. Equity returns come from both price appreciation and capital distributions in the form of dividends. Over long periods, capital appreciation has historically been the dominant contributor to total equity returns, although dividends remain an important source of investor income. Companies differ in dividend policy. Growth-oriented firms may reinvest earnings rather than distribute dividends, whereas mature firms often return a substantial portion of profits to shareholders.

Example 1 Equity Return

An investor buys a stock at \$45. During the next year the price rises to \$52, and the investor receives a dividend of \$1.20. The investor's return can be calculated as:

- Price return = (52 − 45) / 45 = 0.1556 = 15.56%
- Capital distribution return = 1.20 / 45 = 0.0267 = 2.67%
- Total return = 0.1556 + 0.0267 = 0.1823 = 18.23%

Bonds are contractual lending arrangements for which investors receive periodic coupon payments and the return of principal at maturity. Bond returns typically consist largely of capital distributions (coupon income), especially for short-term bonds. However, bond prices fluctuate in response to interest rate changes and credit risk, creating price returns that can either enhance or reduce total return if the bonds are sold prior to maturity. Longer-maturity bonds tend to exhibit greater price variability due to their distant cash flows being more sensitive to interest rate changes.

Example 2 Bond Return

An investor buys a bond at \$95. During the next year the price falls to \$90 and the investor receives a 4.5% coupon. The investor's return can be calculated as:

- Price return = (90 − 95) / 95 = -0.0526 = -5.26%
- Capital distribution return = 4.50 / 100 = 0.045 = 4.50%
- Total return = -0.0526 + 0.045 = -0.0076 = -0.76%

Note that the bond coupon amount (ie, capital distribution) is based on the coupon rate as a percentage of the bond's par value of 100.

Hybrid securities, such as preferred shares and convertible bonds, combine features of both debt and equity. Preferred shares often provide fixed dividends, similarly to bonds, but represent ownership interest. Convertible bonds provide fixed coupon payments but may be converted into equity, introducing equity-like upside potential.

Types of Returns: Financial Instruments

Financial instruments are standardized legal agreements that represent financial assets. These include equities, bonds, securitized products, and derivatives. The return on a financial instrument typically combines price return and capital distribution return. Standardization allows for efficient public trading.

Instruments such as stocks and bonds directly represent ownership or lending relationships. Their total returns are the sum of price appreciation and income distributions. Derivatives differ in that their value is derived from the value of an underlying asset or financial indicator: a derivative's return is driven entirely by changes in its price, which in turn depend on expectations about the underlying asset.

Types of Returns: Financial Indicators

Financial indicators, such as interest rates, exchange rates, and market indexes, do not themselves generate cash flows. However, they influence returns on financial assets and instruments.

Interest rates may be inferred from bond prices and affect both capital distributions and price returns of fixed-income securities. The risk-free rate, often proxied by short-term government debt, serves as a benchmark for evaluating other investments.

Foreign exchange rates represent the value of one currency relative to another. When investors hold foreign assets, total return depends not only on the asset's return in local currency but also on exchange rate movements. Currency appreciation or depreciation can amplify or offset local returns.

Example 3 Currency Effects on Return

A US investor invests USD 500,000 in a European stock fund, when the EUR/USD = 0.80 (ie, USD 1 buys EUR 0.80), so the initial investment is EUR 400,000. While the investors owns the fund, it makes no distributions. The investor sells the fund for EUR 420,000, when the EUR/USD = 0.85.

In EUR, the investor's price return = [(420,000 / 400,000) – 1] = 5%.

In USD, the investor's price return is calculated as:

- USD ending value:
  - 420,000 / 0.85 ≈ $494,118
- USD price return:
  - (494,118 / 500,000) − 1 ≈ −0.0118 ≈ −1.18%

While the investor held the bond, the USD appreciated against the EUR, so the USD return < EUR return on the bond. In this case, the appreciation of the USD was great enough so that the USD price return was negative, even though the fund itself (in EUR) increased in value.

Market indexes aggregate the performance of multiple securities and serve as benchmarks. Investors may gain exposure to indexes through exchange-traded funds (ETFs), mutual funds, or derivatives. Index returns combine price and income components of the underlying securities.

Long-Term Financial Asset Returns

This section provides historical perspective in order to evaluate and compare asset returns relative to their own history and to each other.

Over long periods, equity returns have historically exceeded bond returns, primarily due to capital appreciation. Bonds, by contrast, have generated most of their return from capital distributions (ie, interest income). Short-term Treasury bills have historically produced returns only slightly above inflation, resulting in modest real returns.

The relative contributions of price appreciation and capital distributions vary across decades. Equity returns in some periods have been driven largely by price gains, while in other periods dividend income has played a more prominent role. For bonds, coupon income has historically been the dominant contributor to total return, although periods of rising interest rates can produce capital losses.

Arithmetic average returns typically exceed geometric average returns since arithmetic means do not account as fully for volatility. Geometric returns provide a more accurate measure of compound growth over time.

Example 4 Arithmetic vs. Geometric Returns

An investment has the following annual returns over three years:

Year 1	20%
Year 2	−10%
Year 3	15%

The investment's arithmetic return is calculated as:

$$\frac{0.2 + (-0.10) + 0.15}{3} \approx 0.0833 \approx 8.3\%$$

The investment's geometric return is calculated as:

$$[(1 + 0.20) \times (1 - 0.10) \times (1 + 0.15)]^{1/3} - 1 \approx 0.0749 \approx 7.49\%$$

Risks and Returns

Risk reflects uncertainty about realized returns relative to expected returns. Risks can be divided into systematic (macro) and unsystematic (micro) components.

Systematic risks affect the entire market and include

- inflation risk,
- interest rate risk, and
- market risk.

These risks cannot be eliminated through diversification.

Unsystematic risks are issuer-specific or sector-specific and can be mitigated through diversification. Examples of issuer-specific risk include

- business risk due to the nature of the operations,
- financial risk due to use of leverage,
- sector risk, and
- industry risk.

Historically, higher-risk assets have provided higher average returns as compensation for bearing greater uncertainty. The compensation of return for risk is known as the **risk premium**. However, realized returns may deviate from historical averages, and higher variability does not guarantee higher returns in every period.

Summary: Types of Financial Returns

Financial returns are created from price changes and capital distributions. Equities typically derive most long-term returns from price appreciation, while bonds generate most returns from coupon income. Financial instruments combine these components in varying proportions, and derivatives primarily generate price returns. Financial indicators such as interest rates and exchange rates influence asset returns but do not directly produce cash flows. Historically, higher-risk assets have delivered higher average returns, reflecting the fundamental risk–return trade-off that underlies investment decision making.

Learning Module 3
Benchmarking Returns

LOS: Calculate and compare money-weighted and time-weighted rates of return.

LOS: Describe the choices and the implications of the different weighting methods used in index construction and management, and calculate, interpret, and explain the value and the returns of an index.

Introduction: Benchmarking Returns

While a return can be measured absolutely, its value is relative. This learning module discusses and expands on the tools used to calculate different return measures and the benchmarks they are compared to. It takes a closer look at the methods and measures used to compose a benchmark index, which lends insight into which kind of benchmark is an appropriate comparison for a specific investment.

Money-Weighted and Time-Weighted Rates of Return

LOS: Calculate and compare money-weighted and time-weighted rates of return.

The way we measure investment performance depends critically on how we treat cash flows. When capital is added to or withdrawn from a portfolio, the timing and size of those flows can significantly affect the realized return. Since investment managers and investors often have different levels of control over cash flows, two distinct return measures are used:

- the time-weighted rate of return (TWR) and
- the money-weighted rate of return (MWR), also called the dollar-weighted return or internal rate of return (IRR).

To understand why these measures differ, consider two portfolios that experience identical annual returns but receive contributions at different times. If a large contribution is made just before a strong positive return, the investor's realized return will be high. If that same contribution is made just before a negative return, the investor's realized return will be worse—even though the underlying sequence of returns is identical. Thus, cash flow timing can magnify or dampen the effect of returns.

Investment managers in public markets typically do not control investor contributions or withdrawals. Their performance should therefore be evaluated independently of external cash flow timing. In contrast, private equity managers often control capital calls and distributions, making cash flow timing part of their investment skill.

The Time-Weighted Rate of Return

The time-weighted rate of return measures the compound growth rate of one unit of currency invested at the beginning of the period. It eliminates the effect of external cash flows by dividing the total evaluation period into subperiods defined by cash flow events. A holding period return is calculated for each subperiod, and these returns are geometrically linked.

Formally, the time-weighted return is:

$$r_{TW} = \prod_{t=1}^{T}(1 + r_t) - 1$$

where each subperiod return reflects performance between two cash flow events.

Since it neutralizes the impact of contributions and withdrawals, the time-weighted return isolates the manager's investment performance, regardless of investor decisions about cash flows, since it relies on investment performance between cash flows. It answers the question: *How well did the portfolio perform, on average, per dollar invested at the beginning of the period?*

Example 1 Calculating a time-weighted return

Suppose a portfolio experiences the following annual returns over three years:

Year 1: −20%

Year 2: +15%

Year 3: +10%

The time-weighted return is calculated by geometrically linking the annual returns:

$$[(1 - 0.20)(1 + 0.15)(1 + 0.10)]^{1/3} - 1 = 1.012 - 1 \approx 0.4\%$$

Although the arithmetic average return is 1.67%, the true compound growth rate is only 0.4%. Importantly, this result would be the same no matter when investors contributed or withdrew capital.

For performance evaluation of public equity and fixed-income managers, the time-weighted return is the appropriate benchmark under the Global Investment Performance Standards (GIPS®), as it reflects only what the manager controls: investment decisions.

The Money-Weighted Rate of Return

The money-weighted rate of return incorporates both the timing and magnitude of cash flows. It is defined as the IRR that sets the net present value of all cash flows equal to zero:

$$0 = \sum_{t=0}^{T} \frac{CF_t}{(1 + r_{MW})^t}$$

From the investor's perspective, contributions are negative cash flows and withdrawals or ending values are positive cash flows.

The money-weighted return answers the question: *What was the investor's actual return earned on the invested capital?* Since the money-weighted return metric reflects the timing of capital flows, it is sensitive to investor behavior.

Example 2 Calculating a money-weighted return

An investor contributes:

- €5,000 at Time 0
- €3,000 at the end of Year 1

At the end of Year 2, the portfolio is worth €9,500.

The cash flows from the investor's perspective are:

$CF_0 = -5{,}000$

$CF_1 = -3{,}000$

$CF_2 = +9{,}500$

The money-weighted return is the IRR where:

$$0 = -5,000 + \frac{-3,000}{(1+r)} + \frac{9,500}{(1+r)^2}$$

The IRR can be derived using a financial calculator or spreadsheet and is approximately 11.1%.

This return reflects not only investment performance but also the fact that more capital was invested after the first year. If the second contribution had been made before a year of strong performance, the IRR (ie, the money-weighted return) would be greater; if it were made before a year of weak performance, the IRR would be less.

Comparison of Time-Weighted and Money-Weighted Returns

The key difference between the two measures lies in their treatment of cash flows.

Exhibit 1

Money-weighted rate of return (MWR)	Time-weighted rate of return (TWR)
• Internal rate of return (IRR) of a portfolio, accounts for all contributions and withdrawals • Appropriate for measuring an investor's change in wealth • Affected by amount and timing of cash flows	• Compound growth rate (geometric mean) of a portfolio, excludes contributions and withdrawals • Preferred return measure for evaluating investment performance • Unaffected by amount and timing of cash flows

The time-weighted return excludes the effect of contributions and withdrawals and measures pure investment performance. It is appropriate when the manager does not control external cash flows.

The money-weighted return incorporates the size and timing of cash flows and measures the investor's actual return. It is appropriate when the manager controls capital flows (eg, private equity) or when evaluating the investor experience.

All else equal, if returns are strong early and large contributions are made later, the money-weighted return may be less than the time-weighted return. Conversely, if large contributions precede strong returns, the money-weighted return may exceed the time-weighted return.

Daily Valuation and Compounding

In practice, portfolios are often valued daily. Daily holding period returns are calculated and then geometrically linked to produce the annual time-weighted return:

$$r_{TW} = (1 + r_1)(1 + r_2)...(1 + r_{365}) - 1$$

If using trading days, 252 days is common. Compounding frequency matters. For example, an average daily return of 0.05% produces:

Using 365 days:

$$(1.0005)^{365} - 1 = 20.02\%$$

Using 252 trading days:

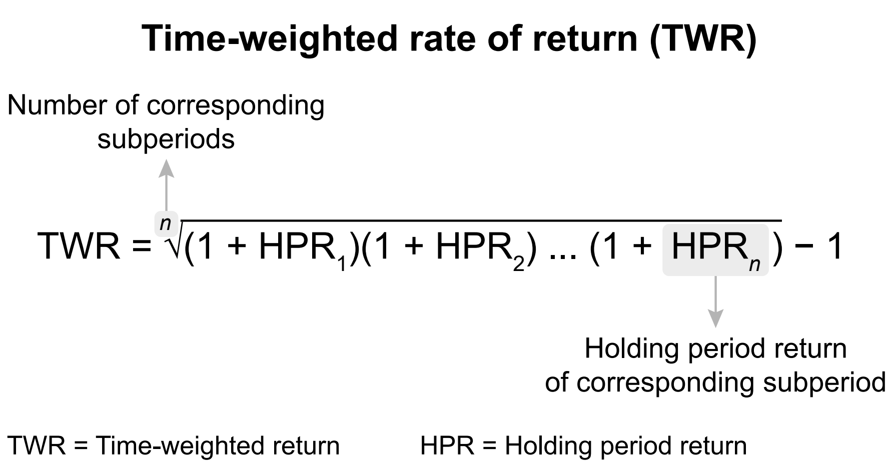
$$(1.0005)^{252} - 1 = 13.42\%$$

Thus, compounding assumptions influence reported returns.

Multi-Year Time-Weighted Returns

For multiple years, annual time-weighted returns are first calculated separately and then annualized using the geometric mean:

Exhibit 2



This ensures consistency with compound growth principles.

Example 3 When returns diverge

A fund earns:

Year 1: +25%

Year 2: −10%

An investor contributes $10,000 at inception and an additional $50,000 at the start of Year 2.

The time-weighted return equals:

$$(1.25)(0.90) - 1 = 12.5\%$$

The compound annual return is:

$$[(1.25)(0.90)]^{0.5} - 1 \approx 6.06\%$$

However, most of the investor's capital was invested before the negative year, so the money-weighted return will be significantly lower: possibly close to zero or negative depending on ending value.

Summary

Evaluating performance, as measured by a rate of return, requires knowing whether:

- arithmetic or geometric returns are used.
- returns are time-weighted or money-weighted.

The time-weighted rate of return removes the impact of external cash flows and measures the intrinsic performance of the portfolio. It is the preferred measure for benchmarking public market investment managers and is the standard benchmark measure under GIPS.

The money-weighted rate of return incorporates the size and timing of cash flows and measures the investor's actual realized return. It is appropriate when managers control capital flows or when evaluating investor experience. For private equity managers, money-weighted returns are often more appropriate since capital calls and distributions are under managerial control. For investors, the money-weighted return best reflects personal investment experience as it accounts for their contribution and withdrawal decisions.

Index Definition and Calculation

LOS: Describe the choices and the implications of the different weighting methods used in index construction and management, and calculate, interpret, and explain the value and the returns of an index.

A security market index is a financial indicator constructed to measure the performance of a group of securities. An index may represent an entire market, a specific region, a sector, an asset class, or a particular investment theme. Conceptually, an index is a portfolio of securities whose value is calculated using observable market prices or estimated values of its constituents.

Indexes serve multiple roles. Historically, they were developed as benchmarks to summarize market performance. Over time, they became the foundation for passive investing strategies, in which portfolios are constructed to replicate index performance rather than attempt to outperform it. Today, indexes guide trillions of dollars in index funds and exchange-traded funds (ETFs), making them both performance measures and investable blueprints.

Exhibit 3

Uses of market indexes

- Indicators of market sentiment
- Proxies for modeling risk, return, or risk-adjusted performance
- Benchmarks for actively managed portfolios
- Proxies for asset classes in asset allocation models
- Model portfolios for investment products

The methodology used to construct an index significantly influences its behavior. Two indexes with identical constituent securities but different weighting schemes can produce meaningfully different returns. Therefore, understanding index construction is essential for performance evaluation, benchmarking, and portfolio management.

Index construction typically involves five steps:

- define the objective,
- select constituents,
- choose a weighting methodology,
- calculate the index value, and
- maintain the index through rebalancing and reconstitution.

The weighting methodology is especially important since it determines how changes in security prices affect overall index performance.

Price Return and Total Return Indexes

Indexes are commonly reported in two forms: price return indexes and total return indexes. A price return index reflects only capital appreciation; it measures percentage changes in constituent prices. A total return index includes both price changes and reinvested capital distributions such as dividends and interest.

At inception, price and total return versions of an index have the same value. Over time, however, the total return index typically exceeds the price return index due to the compounding of reinvested dividends. This divergence grows larger over longer horizons.

Price return indexes are appropriate when benchmarking non-dividend-paying securities or analyzing short-term price trends. Total return indexes are appropriate for evaluating long-term performance, dividend-paying equities, mutual funds, and strategies that reinvest income.

Example 4 Price return vs. total return

Suppose an index consists of a single stock priced at 100. At year-end, the price rises to 108 and the stock pays a dividend of 4.

The price return is:

$$\frac{108}{100} - 1 = 8\%$$

The total return is:

$$\frac{108 + 4}{100} - 1 = 12\%$$

The 4% difference reflects reinvested dividends. Over multiple periods, this compounding effect can be substantial.

Index Weightings

An index's weighting method determines how price changes affect index value. Common weighting schemes include:

- market capitalization weighting,
- float-adjusted market capitalization weighting,
- fundamental-factor weighting,
- price weighting, and
- equal weighting.

Each approach produces different exposures, performance characteristics, and rebalancing requirements.

Market Capitalization-Weighted Index

In a market capitalization-weighted index, each security's weight is proportional to its market capitalization (Price × Shares outstanding). For example, the S&P 500 is a market-cap weighted index. Larger companies therefore have greater influence on index performance.

Exhibit 4

Weight of asset in a market-capitalization-weighted index

Number of shares outstanding

Price of asset

$$\frac{Q_i P_i}{\underbrace{Q_1 P_1}_{\text{Market capitalization of Asset 1}} + Q_2 P_2 + \dots + Q_N P_N}$$

This method is widely used as it reflects the economic size of companies, has relatively low turnover, and rebalances automatically as prices change. If a company's price rises, its weight increases naturally.

The index value equals the total market capitalization divided by a divisor that sets a convenient base value (such as 100 or 1,000). The divisor is adjusted when structural changes occur, to maintain continuity.

A key advantage of this method is its practicality for passive investing. Disadvantages are that index value can become concentrated in large firms and that securities experiencing price momentum become overweighted.

Example 5 Market-cap weighting

Suppose an index contains two companies:

- Company A: 1 million shares at 50 → market cap = 50 million
- Company B: 500,000 shares at 100 → market cap = 50 million

Total market capitalization is 100 million. Each company receives a 50% weight.

If Company B's price increases to 120, its market cap becomes 60 million. Total market cap becomes 110 million, and its weight increases automatically to 54.5%. No rebalancing is required.

Float-Adjusted Market Capitalization-Weighted Index

Not all shares outstanding are available for trading. Some may be held by insiders, governments, or controlling shareholders. Float-adjusted weighting accounts only for shares available to public investors.

This method provides a more accurate representation of investable market opportunities and avoids distortion from large nontradable holdings. Most major global indexes use float-adjusted market capitalization weighting.

Fundamental Factor-Weighted Index

Fundamental weighting assigns weights based on economic metrics such as earnings, revenues, book value, or dividends rather than market capitalization. The weight of each security is proportional to its fundamental measure relative to the total across all constituents.

This approach reflects economic size rather than market price and is often associated with value-oriented strategies. It may overweight companies with strong fundamentals relative to their market valuations.

Fundamental indexes require periodic rebalancing due to price changes that alter the relationship between market value and fundamental measures. If a firm experiences strong price appreciation but earnings remain unchanged, its index weight will not increase until the next rebalance. This differs from a market-cap index, in which price increases in a security immediately raise the weight of that security in the index.

Price-Weighted Index

In a price-weighted index, weights are proportional to stock prices.

$$w_i = \frac{p_n}{p_1 + p_2 + \ldots + p_n}$$

$$p_n = \text{Price of security } n$$

The index value is calculated as the sum of prices divided by a divisor.

Exhibit 5

Value of a price-weighted index

$$V_{\text{PW Index}} = \frac{\overset{\text{Price of security } i}{p_1} + p_2 + \dots + p_n}{\underset{\text{Divisor}}{d}}$$

Higher-priced stocks receive larger weights regardless of company size. This approach is simple but economically less meaningful since it ignores market capitalization. Stock splits require divisor adjustments to prevent artificial changes in index value.

Equal-Weighted Index

An equal-weighted index assigns identical weights to all constituents. Each constituent initially receives a weight of $1/N$. The value of the index is the arithmetic average of its constituents' returns.

Equal-weighted indexes require periodic rebalancing to restore equal allocation. This process results in selling outperformers and buying underperformers. Equal weighting reduces concentration risk but increases turnover and transaction costs.

Example 6 Equal weight vs. market cap

Consider two stocks:

- Large-cap stock rises 5%
- Small-cap stock rises 20%

In a market-cap index where the large stock has 80% weight, the return is:

$$0.80 \times 5\% + 0.20 \times 20\% = 8\%$$

In an equal-weighted index, the return is:

$$0.50 \times 5\% + 0.50 \times 20\% = 12.5\%$$

The equal-weighted index benefits more from small-cap outperformance.

Index Management

Index management includes rebalancing and reconstitution.

Exhibit 6

Reconstitution vs. rebalancing

Reconstitution	Reapply the criteria for securities' inclusion in the index
Rebalancing	Reapply the weighting method to adjust the weights of index securities

Rebalancing adjusts weights to maintain the intended weighting scheme. Equal-weighted and fundamental indexes require scheduled rebalancing. Market-cap indexes rebalance automatically with price movements.

Reconstitution updates index membership. Companies may be added or removed based on eligibility criteria such as market capitalization, liquidity, or sector classification. The divisor is adjusted as needed to maintain index continuity. Reconstitution can affect security prices since funds tracking the index must adjust their holdings.

Comparing Index Performance

Different weighting methods produce different return profiles even when constituents are identical. Price-weighted indexes emphasize high-priced stocks. Equal-weighted indexes reflect average performance and may tilt toward smaller firms. Market-cap indexes are dominated by large companies. Price return and total return versions also diverge over time due to dividend reinvestment.

Exhibit 7

Index weighting methodologies disadvantages

	Company weight in index	Disadvantages
Price	$\frac{\text{Price per share}}{\text{Sum of index share prices}}$	• Arbitrary weights versus other methodologies • Arbitrary reweighting for splits
Equal	$\frac{1}{\text{Number of companies}}$	• Over- (under-) states return impact of small- (large-) caps • Requires frequent rebalancing
Market-cap	$\frac{\text{Company market cap}}{\text{Sum of index company market caps}}$	• Momentum effect ◦ Increases impact of rising stocks ◦ Reduces impact of falling stocks
Fundamental	$\frac{\text{Business scale metric(s)}}{\text{Sum of metric(s) for index companies}}$	• Value tilt • Contrarian effect (opposite of momentum effect)

Understanding these differences is critical when benchmarking performance. A manager should be evaluated against a benchmark consistent with the strategy's investment universe and weighting structure.

Summary

An index is a constructed portfolio used for measurement and benchmarking. Weighting methodology significantly affects index behavior, risk exposure, and performance.

- Market-cap weighting reflects economic size and requires minimal turnover.
- Equal weighting reduces concentration but requires frequent rebalancing.
- Fundamental weighting emphasizes economic metrics rather than prices.
- Price weighting is simple but economically limited.

The distinction between price return and total return indexes is crucial for long-term performance evaluation. Proper benchmarking requires alignment between portfolio strategy and index construction.